CORRIGENDUM TO: THE LEVEL FOUR BRAID GROUP
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Abstract. The proof of the first statement of Theorem 5.1 of the paper referenced in the title is correct for \( k = 1 \) and incorrect for \( k \geq 2 \) and should be considered an open problem. As such, the proof of the second statement is not correct for \( k \geq 2 \).

This note is an erratum for the published version of our paper [2]. The arXiv has been updated with the corrections described here. As in our paper, let \( \rho \) be the symplectic representation of \( B_n \), let \( \pi_1(D'_n, p_1), \ldots, \pi_1(D'_n, p_n) \) denote the point pushing subgroups of \( B_n \), and for \( 1 \leq k \leq n \) set

\[
K_{n,k} = \pi_1(D'_n, p_1) \cap \cdots \cap \pi_1(D'_n, p_k)
\]

Also, let \( \Gamma_n[m] \) denote \( \text{Sp}_{2g} (\mathbb{Z})[m] \) when \( n = 2g + 1 \) and \( (\text{Sp}_{2g+2} (\mathbb{Z})[m] \mid y_{g+1} \) when \( n = 2g + 2 \).

Theorem 5.1 describes \( \rho(K_{n,k}) \) for \( n \geq 5 \). The theorem separately addresses the cases where \( n = 2g + 1 \) and \( n = 2g + 2 \). In each case, there are two statements. The first statement is that \( \rho(K_{n,k}) \) contains \( \Gamma_n[4] \) and the second statement describes the quotient of \( \rho(K_{n,k}) \) by \( \Gamma_n[4] \). We refer to these two statements as the containment statement and the quotient statement, respectively.

The proof of the containment statement of Theorem 5.1 is correct for \( k = 1 \) and incorrect for \( k \geq 2 \). What our argument for the containment statement actually shows is that each \( \rho(\pi_1(D'_n, p_i)) \) contains \( \Gamma_n[4] \) and hence the argument only proves the weaker statement that

\[
L_{n,k} = \rho(\pi_1(D'_n, p_1)) \cap \cdots \cap \rho(\pi_1(D'_n, p_k))
\]

contains \( \Gamma_n[4] \). Since \( L_{n,1} = \rho(K_{n,1}) \), the argument for the containment statement is correct for \( k = 1 \) and \( n \geq 5 \). For \( k \geq 2 \) we have \( L_{n,k} \supseteq \rho(K_{n,k}) \), but this is not an equality in general.

It should be considered an open question as to whether the containment statement of Theorem 5.1 is correct for \( k \geq 2 \). At the end of the paper, we explain how our proof of Theorem 5.1 can be extended to the case \( n = 3 \), in particular that \( \rho(K_{3,k}) \) contains \( \Gamma_3[4] = \text{SL}_2(\mathbb{Z})[4] \). This statement, the \( n = 3 \) version of the containment statement, is not correct. In particular, the last statement in the paper, that \( \rho(K_{3,3}) = \Gamma_3[4] \), is not correct. In fact, \( \rho(K_{3,3}) \) has infinite index in \( \text{SL}_2(\mathbb{Z}) \). To see this, we first note that \( K_{3,3} \) is the Brunnian subgroup of \( B_3 \). Let \( Z \) denote the kernel of \( \rho : B_3 \to \text{SL}_2(\mathbb{Z}) \).
The group $Z$ is an infinite cyclic group generated by the square of the Dehn twist about the boundary of $D'_3$. For $m \neq 0$, no element of the coset $\sigma^m Z$ is Brunnian, hence no power of the matrix $\rho(\sigma_1)$ lies in $\rho(K_{3,3})$.

The statement and proof of the quotient statement of Theorem 5.1 are correct for $k = 1$. Because of the $n = 3$ case, we expect that the containment statement of Theorem 5.1 is not correct for any $k \geq 2$ and $n \geq 5$. If this is the case, the quotient statement does not make sense for $k \geq 2$.

As in the $n = 3$ case, we expect that $\rho(K_{n,k})$ in fact has infinite index in $\Gamma_n[4]$ for $n \geq 4$ and $k \geq 2$. As in the $n = 3$ case, the $k = n$ version of this statement can be proven by showing that if $h \in \ker(\rho)$ then $\sigma^m h$ is not Brunnian. Since $\ker(\rho)$ is generated by squares of Dehn twists about curves surrounding an odd number of punctures [1], we may assume that $h$ is such a product.

What our argument for the quotient statement of Theorem 5.1 actually shows is that the image of $\rho(K_{n,k})$ in $\Gamma_n[2]/\Gamma_n[4]$ is $(\mathbb{Z}/2)^2$, $\mathbb{Z}/2$, or 1, according to whether $k$ is 1, 2, or greater. In other words, $\rho(K_{n,k})$ modulo $\rho(K_{n,k}) \cap \Gamma_n[4]$ is the abelian group given in the previous sentence. It is also true that $L_{n,k}/\Gamma_n[4]$ is the same abelian group. The given indices of $\rho(K_{n,k})$ in $\Gamma_n[2]$ for $k \geq 2$ are the correct indices for $L_{n,k}$ in $\Gamma_n[2]$.

There are two other incorrect statements in Section 3 of the published paper that we would like to point out. First, we incorrectly state that $\rho(B_n)$ is the semi-direct product of a symmetric group with $\Gamma_n[2]$. In fact $\rho(B_n)$ is a non-split extension of these groups. Also, we incorrectly state that $\psi : \text{Sp}_{2g}(\mathbb{Z}/2) \to \text{sp}_{2g}(\mathbb{Z}/2)$ is the abelianization map for $\text{Sp}_{2g}(\mathbb{Z}/2)$ (Sato proved that the abelianization is larger [3, Corollary 10.2]). We are grateful to David Benson and Nick Salter for these corrections.

References

