More Free Groups in MCG

We showed: $f_1, f_2 \in MCG$ pA $\rightsquigarrow \exists\, n$ s.t. $\langle f_1^n, f_2^n \rangle$ is abelian or free.
That proof generalizes to $f_1, \ldots, f_k$ pA.
Want to generalize in two more ways: ① f_i are partial pA
② $k = \infty$.

First...

More free groups in $\mathrm{Isom}(\mathbb{H}^2)$

Say $a, b \in \mathrm{Isom}(\mathbb{H}^2)$ parabolic.
WTS $\exists\, n$ s.t. $\langle a^n, b^n \rangle \cong F_2$.

Key is "BGI": If A, B, C are horoballs with $d(\pi_C(A), \pi_C(B)) > M$
then the geodesic from A to B passes thru C.

Choose horoballs A, B preserved by a, b and distance 1 apart.
Replace a, b with powers s.t. $d_A(B, aB) \geq 2M$
$d_B(A, bA) \geq 2M$
Create an "electrified space" by coning off each horoball
in the $\langle a, b \rangle$-orbit of A, B.

Let $\omega = a^{p_1} b^{p_2} \cdots a/b^{p_L} \in \langle a, b \rangle$
$= s_1 \cdots s_L$

To show: $d(\omega(B), B) \geq L$ in electrified space

$\Rightarrow \omega \neq id \Rightarrow \langle a, b \rangle \cong F_2$.

Let $B_i = s_1 \cdots s_i (B)$ i odd

$\quad\;\; = s_1 \cdots s_i (A)$ i even

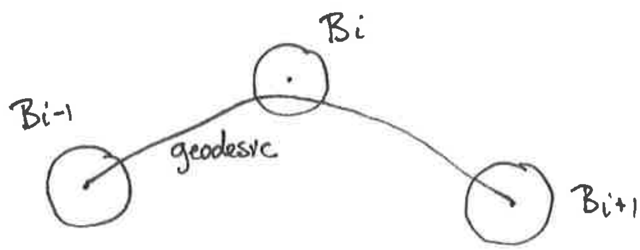
and $B_{-1} = B$.

Claim. $d_{B_i}(B_{i-1}, B_{i+1}) \geqslant 2M$ (dist of proj's)

Pf. Say i odd.

$$\begin{aligned} d_{B_i}(B_{i-1}, B_{i+1}) &= d_{s_1 \cdots s_i (B)}(s_1 \cdots s_{i-1}(A), s_1 \cdots s_{i+1}(A)) \\ &= d_B(s_i^{-1}(A), s_{i+1}(A)) \\ &= d_B(A, s_{i+1}(A)) = d_B(A, b^k A) \\ &\geqslant 2M \end{aligned}$$

By BGI have this picture:



Want to string these together: if the geodesic from B_0 to B_L passes through all B_i, the distance is at least L.

Assume by induction that any geodesic from B_0 to B_{k-1} passes through $B_0, \ldots, B_{k-1}$.

Claim. $\exists$ geodesic from B_0 to B_{k-2} avoiding B_{k-1}

Pf. Say γ from B_0 to B_{k-2} passes in B_{k-1}.

By induction the intial segment from B_0 to B_{k-1}

passes thru $B_{k-2} \rightsquigarrow \gamma$ can be shortened.

(use the coning off!)

By Claim and BGI, $d_{B_{k-1}}(B_0, B_{k-2}) \leq M$

Now: $d_{B_{k-1}}(B_0, B_k) \geq d_{B_{k-1}}(B_{k-2}, B_k) - d_{B_{k-1}}(B_0, B_{k-2})$

$$\geq 2M - M$$
$$= M$$

By BGI any geod from B_0 to B_k passes thru B_{k-1}

And by induction such a geod passes thru $B_0, \ldots, B_k$

To conclude $d(B_0, B_L) \geq L$ remains to show the B_i are pairwise disjoint. Suppose $z \in B_i \cap B_{i+k}$. By the above, the constant geodisic z passes thru $B_i, \ldots, B_{i+k} \Rightarrow z \in B_i \cap B_{i+1}$, a contradiction. ▨

To Do: ① Redo the argument without coning. Instead use Behrstock inequality. (see email from Mangahas 11/12/14)

② Show all elements of $\langle a, b \rangle$ not conj to power of generator are hyperbolic isometries. Key: parabolics/elliptics move pts sublinearly.

Free Groups from Partial Pseudo-Anosovs (Mangahas)

Simple case. $A, B \subseteq S$

$\alpha = \partial A$, $\beta = \partial B$ ← $\partial A, \partial B$ conn.

$d_{C(S)}(\alpha, \beta) \geq 3$.

a, b partial pAs supp. on A, B.

Basically the same argument. Need to say what horoballs are:

$$C_A = \{ v \in C(S) : \pi_A(v) = \emptyset \} \subseteq N_1(\alpha)$$

similar $\quad C_B \subseteq N_1(\beta)$

Note: $d(\alpha, \beta) \geq 3 \implies C_A \cap C_B = \emptyset$.

Replace a, b with high powers s.t.

$d_A(C_B, a(C_B)) \geq 2M+4$ ← d_A means diam of union of two proj's.

$d_B(C_A, b(C_A)) \geq 2M+4$

First one implies: $d_A(v, a^k(v')) \geq 2M \quad \forall\, v, v' \in C_B$.

since diam $C_B = 2$.

etc. Just run through the same argument.

Since pA's are only elts with unbounded orbits, immediately get that all elements of $\langle a, b \rangle$ not conj to a power of a or b is pA.

BEHRSTOCK LEMMA

$\xi(S)$ = complexity $= 3g-3+n =$ $\dim C(S)+1$.

<u>Lemma.</u> $Y, Z \subseteq S$ overlapping
$\xi(Y), \xi(Z) \geq 4$.
x = curve with $\pi_Y(x), \pi_Z(x) \neq \emptyset$.
Then $\qquad d_Y(x, \partial Z) \geq 10 \Longrightarrow d_Z(x, \partial Y) \leq 4$

i.e. can't both be large.
This is analogous to Fact 3 above. (think of x as ∂X).

<u>Facts.</u> Let $U \subseteq S \qquad \xi(U), \xi(S) \geq 4$.
$u, v \in C(S)$
a_u, a_v projection arcs in U
$\pi_U(u), \pi_U(v)$ projection curves.

① $i(a_u, a_v) = 0 \Rightarrow d_U(u,v) \leq 4$
② $i(u,v) > 0 \Rightarrow i(u,v) \geq 2^{(d_U(u,v)-2)/2}$
③ $i(u,v) \leq 2 + 4 \cdot i(a_u, a_v)$.

Pf of <u>Lemma</u> (Leininger). $\quad d_Y(x, \partial Z) \geq 10 > 2 \Rightarrow$ distance realized by curves $u \in \pi_Y(x)$, $v \in \pi_Y(\partial Z)$ s.t. $i(u,v) \geq 2^4 = 16$ (Fact ②). Now, u & v come from arcs a_u, a_v with $i(a_u, a_v) \geq (16-2)/4 > 3$ (Fact ③). Note $a_u \subseteq x$, $a_v \subseteq \partial Z$. One arc of a_u b/w pts of intersection with a_v lies in Z. This arc is disjoint from x-arcs in Z, so $d_Z(x, \partial Y) \leq 4$ (Fact 1). $\blacksquare$

Free Groups via Ping Pong (Manganas à la Ishida & Hamidi-Tehrani)

a, b pA with supports A, B

$\xi(A), \xi(B) \geq 4$

$A \cap B \neq \emptyset$.

Choose n s.t. translation distance of a on $\mathcal{C}A(S)$ is ≥ 14
and same for b.

<u>Prop.</u> $\langle a^n, b^n \rangle \cong F_2$

<u>Pf.</u> Ping pong

needed?

$$X_a = \{ v : \pi_A(v), \pi_B(v) \neq 0 \ , \ d_A(v, \partial B) \geq 10 \}$$

X_b similar. Note $X_a \cap X_b = \emptyset$ by Behrstock.

Take $v \in X_a$.

Behrstock $\Rightarrow d_B(v, \partial A) \leq 4$

$\Rightarrow d_B(b^n(v), \partial A) \geq 10$

$\Rightarrow b^n(v) \in X_b$ ∎

Broad outline of proof. First we cone off the $Q_i \subseteq X$ and show result is δ-hyp

(use: fellow traveller condition)

The R_i now rotate about cone points

moving family $\rightsquigarrow$ rotating family

large inj rad $\rightsquigarrow$ very rotating : if we take a pt x sufficiently far from a cone pt c, then rotate about c by g then the geodesic from x to gx passes thru c (like BG1).

In this sense, the proof is reminiscent of last lecture.

Windmills. A windmill is a subset $W \subseteq X$ with

① W almost convex

② $N_{40\delta}(W) \cap C = W \cap C \neq \emptyset$ $\qquad C$ = set of cone pts

③ $G_W = \langle G_c : c \in W \cap C \rangle$ preserves W $\qquad G_c$ = rotating elt

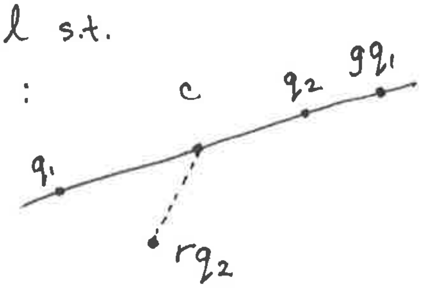
④ $\exists\ S_W \subseteq W \cap C$ s.t. $G_W \cong \ast_{c \in S_W} G_c$

⑤ (Greendlinger condition) Every elliptic in G_W lies in some G_c, $c \in S_W$. Other elts have invar. geod. axis ℓ s.t. $\ell \cap C$ contains at least 2 g-orbits of pts at which there is a shortening elt

Shortening elt ℓ = axis for g, contains $c \in C$

shortening elt is $r \in G_c \setminus id$ s.t. $\exists\ q_1, q_2 \in \ell$ s.t. $d(q_1, q_2) \in [24\delta, 50\delta]$ but $d(q_1, rq_2) \leq 20\delta$:

Triangle $\leq$ $\Rightarrow$ rg has shorter transl. length than g.



Infinitely Generated Free Groups

Thm (Dahmani–Guirardel–Osin) $f \in MCG(S)$ pA.
$\exists\, n$ s.t. $\langle\langle f^n \rangle\rangle \cong F_\infty$
and all nontrivial elements pA.

Inspired by:

Thm (Gromov) $\exists\, m = m(k,\delta)$ s.t. if $\gamma_1, \dots, \gamma_k$ are hyp. elements of a δ-hyp gp the normal closure of the $\gamma_i^{m_i}$ is free when $m_i \geq m \ \forall i$.

Aside: Whittlesey's groups

$F_i : MCG(S_{0,n}) \to MCG(S_{0,n-1})$ forget i^{th} marked pt
$\text{Brun}(S_{0,n}) = \bigcap \ker F_i$ "Brunnian"

A Brunnian braid

Thm. For $n \geq 5$ $\text{Brun}(S_{0,n})$ is all pA (it is obviously normal).

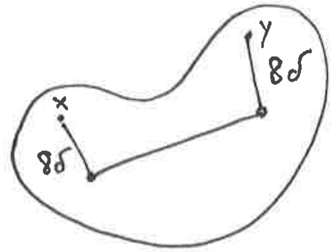
Pf. By NT Classification, suffices to rule out periodic, reducible.
Easy to rule out periodic, either by Birman exact seq or classification of torsion in $MCG(S_{0,n})$.
Say an elt f of $\text{Brun}(S_{0,n})$ has a reducing curve c.
On one side of c, f is doing something nontrivial.
Forget a marked pt on the other side $\rightsquigarrow F_i(f) \neq \text{id}$. ∎

Small Cancellation Theory.

$X = \delta$-hyp space

$G \circlearrowright X$ by isoms.

$(Q_i)_{i \in I}$ almost-convex subspaces : $\forall\, x, y$



(think: axes)

$(R_i)_{i \in I}$ $R_i \triangleleft \mathrm{Stab}_G Q_i$

(think: hyp. elts)

$G \circlearrowright I$ with $Q_{gi} = gQ_i$

$R_{gi} = gR_ig^{-1}$

$\mathcal{F} = \{(Q_i), (R_i)\}$ "moving family"

Injectivity radius : $\mathrm{inj}(\mathcal{F}) = \inf\{d(x, gx) : i \in I,\ x \in Q_i,\ g \in R_i \setminus id\}$

Fellow traveling const : $\Delta(Q_i, Q_j) = \mathrm{diam}\ N_{20\delta}(Q_i) \cap N_{20\delta}(Q_j)$

note: $Q_i \setminus$ this intersection is far from Q_j by δ-hyp.

$$\Delta(\mathcal{F}) = \sup_{i \neq j} \Delta(Q_i, Q_j)$$

$\mathcal{F}$ satisfies (A, ε)- small cancellation if

① $\mathrm{inj}(\mathcal{F}) \geq A\delta$

② $\Delta(\mathcal{F}) \leq \varepsilon\, \mathrm{inj}(\mathcal{F})$

<u>Thm</u> (DGO) $\exists\, A_0, \varepsilon_0$ s.t. if $\mathcal{F}$ satisfies (A, ε)-small cancell. with $A \geq A_0$, $\varepsilon \geq \varepsilon_0$ then

$\langle\langle R_i \rangle\rangle$ is a free product on some of the R_i.

<u>Thm</u>: MCG satisfies small canc. with $R_i = f_i^N$, f_i pA Q_i = axes.