MMM Classes

$S_g \to E \to B$

$\rightsquigarrow$ V = vertical 2-plane bundle on E.

$$e_1(E) = \text{Gysin}\left(e(V)^2\right) \in H^4(B).$$

For $B = S_h$ compute by intersecting 2 generic sections with 0-section, since ① e is P. dual to section $\cap$ 0-section

② $\cup$ is P. dual to $\cap$

③ Gysin is P. dual to projection.

We will see: if E_1 diffeo. E_2 then $e_1(E_1) = e_1(E_2)$

e.g. Atiyah-Kodaira: $\begin{array}{ccc} S_4 & \to & M \\ & & \downarrow \\ & & S_{17} \end{array}$ $\quad$ $\begin{array}{ccc} S_{49} & \to & M \\ & & \downarrow \\ & & S_2 \end{array}$

Say e_1 geometric.

More generally: $e_i(E) = \text{Gysin}\left(e(V)^{i+1}\right) \in H^{2i}(B)$

Compute by intersecting $i+1$ sections with 0-section.

Thm. (Church-Farb-Thibault) e_{2i+1} geometric.

Want to show $e_i \neq 0$. Need $S_g \to M^{2i+2} \to B^{2i}$ with $e_i(M) \neq 0$ $\forall g, i$.

Will use branched covers.

SIGNATURE

M = closed, oriented $4k$-manifold

$$\rightsquigarrow \quad H^{2k}(M;\mathbb{Q}) \otimes H^{2k}(M;\mathbb{Q}) \longrightarrow H^{4k}(M;\mathbb{Q}) \approx \mathbb{Q}$$
$$\alpha \otimes \beta \longmapsto \alpha \cup \beta$$

bilin. form, symmetric since $2k$ even.

$\sigma(M)$ = signature of this form : # pos. eigen vals − # neg. eigenvals

Rochlin : $\sigma(M^4) = 0 \Leftrightarrow M^4 = \partial W^5$

Hirzebruch: $p_1(M^4) = 3\sigma(M^4)$ (baby case of H. σ formula)

Prop. $S_g \to E \to S_h$
$$\Rightarrow \langle e_1(E), S_h \rangle = \langle p_1(E), E \rangle \; (= 3\sigma(E))$$

Cor. e_1 is geometric.

Pf of Prop. $TE \cong V \oplus \pi^* S_h$

$$\rightsquigarrow p_1(E) = p_1(V \oplus \pi^* S_h)$$
$$= p_1(V) + \pi^* p_1(S_h)$$
$$= e(V)^2 + 0 \qquad \text{in general } p_1 = e^2$$
$$\Rightarrow \langle e_1(E), S_h \rangle = \langle \mathrm{Gysin}(e(V)^2), S_h \rangle$$
$$= \langle e(V)^2, E \rangle$$
$$= \langle p_1(E), E \rangle$$

exercise:
① $\mathrm{Gysin}(\alpha)(\sigma) = \alpha(\pi^* \sigma)$
② $\pi^* S_h = E$

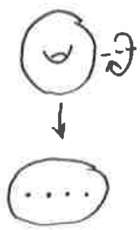
$\blacksquare$

Branched Covers

A cyclic branched cover is a map $\tilde{M} \xrightarrow{p} M$ that is a cyclic covering away from a codim 2 subman of M = ramification locus
(can allow more complicated ram. locus, but we won't)

$\forall\, p \in M\ \exists$ nbd U s.t. $p^{-1}(U) \to U$ is
① trivial m-fold cover (m copies of U), or
② quotient by order m rotation (m = degree of cover)

e.g.



Can sometimes get cyclic branched covers via group actions: Say
$\mathbb{Z}/m \circlearrowright N$ by or. pres. diffeos s.t. ① fixed set has codim 2, F = mnfld
② action free outside F

Then $\bar{N} = N/\mathbb{Z}/m$ is a manifold (check!) and $N \to \bar{N}$ is cyclic b. cover
Near F, proj looks like $F \times \mathbb{C} \to \bar{F} \times \mathbb{C}$

$$(p, z) \mapsto (p, z^m)$$

<u>Thm</u>. Every closed, or. 3-man is a 3-fold simple branched cover over S^3.

Existence of Branched Covers

Prop. M = closed or. smooth n-man.
$B \subseteq M$ or. subman of codim 2.
If $[B] \in H_{n-2}(M)$ divis. by m. in $H_{n-2}(M;\mathbb{Z})$.
then $\exists$ m-fold cyclic ~~cover~~ branched cover over M ramified along B.

Proof for $M = S^3$, $B = K$. Let S = Seifert surface

$$\rightsquigarrow [S] \in H_2(S^3, K) \cong H^1(S^3 - K)$$

(via $H_2(S^3, K) \longrightarrow H_2(S^3 - K, N(K) - K) \rightarrow H_2(S^3 - N(K), \partial N(K))$
$\xrightarrow{P.D.} H^1(S^3 - N(K)) \longrightarrow H^1(S^3 - K)$)

The elt of H^1 is signed intersection with S.
An elt of $H^1(S^3 - K)$ is a map $H_1(S^3 - K) \longrightarrow \mathbb{Z}$.
Reduce mod any m, get a cover over $S^3 - K$.
Glue K into the cover.

This works in general. There is no Seifert surface per se, but there is a class in A $H_{n-1}(M, \mathbb{Z}_m)$ with boundary B. Then, elts of $H^1(M; \mathbb{Z}_m)$ are maps $H_1(M; \mathbb{Z}) \longrightarrow \mathbb{Z}_m$, so can proceed as above.

We know the elt of H^1 is nontrivial by considering a small loop around B in M. It intersects A in one pt.

EXISTENCE OF BRANCHED COVERS II

Vector Bundle Version.

Suppose $[B] = m[A]$ in $H_{n-2}(M;\mathbb{Z})$

Let $[B]^*, [A]^*$ be P. duals.

We know:

$$\begin{array}{l}\text{Group of } \mathbb{C}^1\text{-bundles} \\ \text{on } M \text{ under } \otimes\end{array} \cong H^2(M;\mathbb{Z})$$

Let E_B be $\mathbb{C}^1$-bundle corr. to $[B]^*$. This means

E_B has a section $s: M \to E_B$ s.t.

$$\operatorname{Im}(s) \cap M = B.$$

Similarly, $E_A \leftrightarrow [A]^*$. By above isomorphism:

$$E_A^{\otimes m} \cong E_B$$

Define

$$f: E_A \longrightarrow E_B$$
$$v \longmapsto v \otimes \cdots \otimes v = v^m$$

Set

$$\widetilde{M} = f^{-1}(\operatorname{Im}(s)).$$

Each pt of $M - B$ has m preimages: the mth roots.

Branched Covers and Euler Classes

A cyclic branched cover $\tilde{E} \xrightarrow{p} E$ is a fiberwise cyclic branched cover of surface bundles if the restriction of p to a (surface) fiber is a branched cover of surfaces onto a fiber of E.

Equivalently $\tilde{E}$ is a cyclic branched cover over E s.t. ramification locus intersects each fiber of E in a 0-manifold.

(use: the restriction of a (branched) cover to a subman. of base is a branched cover)

Prop. Let $\tilde{E} \xrightarrow{p} E$ be a 2-fold fiberwise cyclic branched covers over M with fiber genus $2g$ & g. Then

(1) $p^*[D]^* = 2[\tilde{D}]^*$ D = ram. locus.

(2) $e(\tilde{V}) = p^* e(V) - [\tilde{D}]$

Note: (1) is just a fact about branched covers.

Pf of (1). ~~$p^*[D]^*$ computed~~ ~~[illegible]~~. Clear when D is a 0-manifold. In general, replace fundamental class with Thom class of normal bundle.

Pf of (2). Clearly:

$$\begin{tikzcd} H^2(E) \arrow[r, "p^*"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & H^2(\tilde{E}) \arrow[d] \\ H^2(E \setminus \operatorname{Int} N(D)) \arrow[r] & H^2(\tilde{E} \setminus \operatorname{Int} N(\tilde{D})) \end{tikzcd}$$

$N(D)$ = tub. nbd.

(check on the level of bundles).

$\Rightarrow$ $e(V)$, $e(\tilde{V})$ have same image in lower right.

Consider LES of pair:

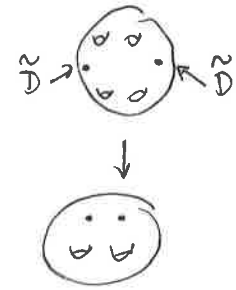
$$\cdots \longrightarrow H^2(\tilde{E}, \tilde{E} \setminus \operatorname{Int} N(\tilde{D})) \longrightarrow H^2(\tilde{E}) \longrightarrow H^2(\tilde{E} \setminus \operatorname{Int} N(\tilde{D})) \longrightarrow \cdots$$

Since $p^* e(V)$, $e(\tilde{V})$ have same image in

they differ by elt of

$$H^2(\tilde{E}, \tilde{E} \setminus \operatorname{Int} N(\tilde{D})) \cong H^2(N(\tilde{D}), \partial N(\tilde{D}))$$
$$\cong H_{n-2}(\tilde{D}) \cong \mathbb{Z}.$$

Remains to compute this integer: Evaluate $p^*(e(V)) + k[\tilde{D}]^*$ and $e(\tilde{V})$ on fiber S_{2g} of $\tilde{E}$:



$$e(\tilde{V})(S_{2g}) = 2 - 2(2g) = 2 - 4g.$$

since fibers map with degree 2. $\longrightarrow$ $p^*(e(V))(S_{2g}) = 2(2-2g) = 4-4g$

$k[\tilde{D}]^*(S_{2g}) = 2k$ $\leftarrow$ $\tilde{D}$ intersects each fiber in 2 pts

$\rightsquigarrow$ $2 - 4g = 4 - 4g + 2k$

$\Rightarrow$ $k = -1$, as desired.

Thm. $\tilde{E} \xrightarrow{p} E$ as above. Then:

$$e_1(\tilde{E}) = 2e_1(E) - 3i(\tilde{D}, \tilde{D})$$

Pf. By Prop (2):

$$e(\tilde{V}) = p^*(e(V)) - [\tilde{D}]^*$$

Squaring:

$$e(\tilde{V})^2 = p^*(e(V)^2) - 2p^*(e(V))[\tilde{D}]^* + [\tilde{D}]^{*2}$$

Use Prop(1) $\longrightarrow$

$$\begin{aligned} e_1(\tilde{E}) &= 2e_1(E) - 2\left(e(\tilde{V})[\tilde{D}]^* + [\tilde{D}]^{*2}\right) + [\tilde{D}]^{*2} \\ &= 2e_1(E) - i(\tilde{D}, \tilde{D}) - 2e(\tilde{V})[\tilde{D}]^* \end{aligned}$$

Remains to show: $e(\tilde{V})[\tilde{D}]^* = i(\tilde{D}, \tilde{D})$.

But since $\tilde{V}$ is transverse to $\tilde{D}$ at all points, its restriction to $\tilde{D}$ is isomorphic to the normal bundle $N\tilde{D}$

$$\Rightarrow \quad \begin{aligned} e(\tilde{V})[\tilde{D}]^* &= e(\tilde{V})(\tilde{D}) \\ &= e(N\tilde{D})(\tilde{D}) \\ &= i(\tilde{D}, \tilde{D}). \end{aligned}$$

$\blacksquare$

Atiyah's Construction

Will form a 2-fold branched cover over $S_{129} \times S_3$.

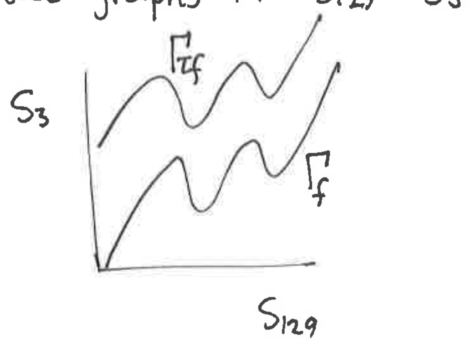
$\rightsquigarrow$ need a D with $[D]$ even.

Start with two covers:

$$\begin{array}{cl} S_{129} & \\ \downarrow f & \text{cover corr. to } \pi_1(S_3) \longrightarrow H_1(S_3; \mathbb{Z}_2) \\ S_3 & \\ \downarrow h & \text{quotient by } \langle \tau \rangle \\ S_2 & \end{array}$$

key: $f^* = 0$ on $H^1(S_3; \mathbb{Z}_2)$
$h^* = 0$ on $H^1(S_2; \mathbb{Z}_2)$

D is union of two graphs in $S_{129} \times S_3$:



"key" will $\Rightarrow$ $[D]$ is even

Some features: ① $\Gamma_f \cap \Gamma_{\tau f} = \emptyset$ since τ has no fixed pts

② Vertical bundle V (= pullback of TS_3 via proj to S_3) is transverse to D

③ Projection $D \longrightarrow S_3$ is a covering map (namely f).

④ Each S_3-fiber intersects D in two pts.

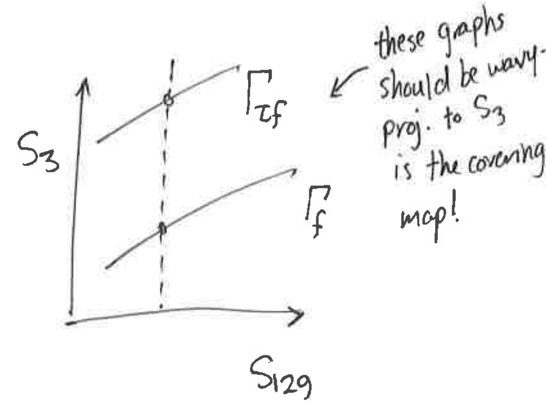
② $\Rightarrow$ $V|_D \cong ND$ normal bundle

③ $\Rightarrow$ $V|_D \cong TD$ tangent bundle.

① $\Rightarrow$ $i(D,D) = 2\,i(\Gamma_f, \Gamma_f)$

④ $\Rightarrow$ when we take the branched cover over D, fibers are S_6.

Claim ① $[D]$ is even.

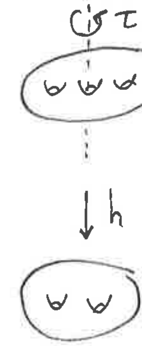


Let $[D]^*$ be P. dual,

$[D]^*_2 \in H^2(S_{129} \times S_3; \mathbb{Z}_2)$

the mod 2 reduction

Need $[D]^*_2 = 0$.



$$\begin{array}{ccccc} D & & & & \Delta \\ \cap & & & & \cap \\ S_{129} \times S_3 & \xrightarrow{f \times id} & S_3 \times S_3 & \xrightarrow{h \times h} & S_2 \times S_2 \end{array}$$

$$[D]^*_2 = (f \times id)^* (h \times h)^* [\Delta]^*_2$$

But $H^2(S_2 \times S_2) \cong H^2(S_2 \times pt) \oplus (H^1(S_2) \otimes H^1(S_2)) \oplus H^2(pt \times S_2)$

and $(h \times h)^*$ kills H^2 factors since h has deg 2

$(f \times id)^*$ kills middle factor since

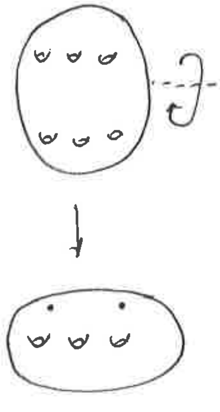
$f_*(H_1(S_{129}; \mathbb{Z})) \subseteq 2H_1(S_3; \mathbb{Z})$ by defn.

Thus $\exists$ 2-fold cyclic branched cover $E \to S_{129} \times S_3$

with ram. locus D.

E has the structure of a surface bundle over S_{129}

Fiber is S_6:



Thm. $e_1(E) = 768 \neq 0$.

Pf. By previous Thm:
$$\begin{aligned} e_1(E) &= 2e_1(S_{129} \times S_3) - 3\, i(\tilde{D}, \tilde{D}) \\ &= -3\, i(\tilde{D}, \tilde{D}) \\ &= -3/2\, i(D, D) \qquad \text{by Prop (1).} \\ &= -3\, i(\Gamma_f, \Gamma_f) \end{aligned}$$

Recall from above that the normal bundle $N\Gamma_f$ is isomorphic to the tangent bundle $T\Gamma_f$ (both are $\cong$ to $V|_{\Gamma_f}$).

So:

$$i(\Gamma_f, \Gamma_f) = e(N\Gamma_f) = e(T\Gamma_f) = \chi(\Gamma_f) = \chi(S_{129}).$$

$\blacksquare$