Math 8803 Homework 1

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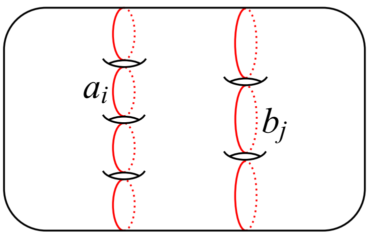
Proposition 1. *The only n-gons in the complex $B_x(S_g)$ have $n = 3, 4, 5, 6$.*

Proof. Let C be a reduced multicurve which represents x with some weight. Since we want 2-cells, we assume $S_g \setminus C$ has 3 components. We can ignore curves which are not part of the boundary of some component of $S_g \setminus C$, since the weights cannot be shifted to other curves.

Consider the dual graph of C. Since C is reduced, the graph is recurrent. If we ignore the directions and multiplicities of edges, the new graph is connected. There are two possibilities.

Case 1: •——•——•. The surface is homeomorphic to the picture below.



Here the curves on the left are labeled $a_1, \cdots, a_n$, not necessarily in order. Similarly, the curves on the right are labeled $b_1, \cdots, b_m$. The components of $S_g \setminus C$ provides two relations

$$\sum_{i=1}^{n} s_i[a_i] = 0, \quad \sum_{j=1}^{m} t_j[b_j] = 0,$$

where $s_i, t_j = \pm 1$ depend on the orientations of the curves. By relabeling the curves, we can assume

$$s_1 = \cdots = s_k = 1, \qquad s_{k+1} = \cdots = s_n = -1,$$
$$t_1 = \cdots = t_l = 1, \qquad t_{l+1} = \cdots = t_m = -1$$

for some $1 \le k < n$ and $1 \le l < m$. This is possible since C is reduced. Let $p = \sum_{i=1}^{n} \alpha_i a_i + \sum_{j=1}^{m} \beta_j b_j$ be a point in the 2-cell. Using the relations, we can eliminate $[a_1]$ and $[b_1]$ in $[p]$ to get

$$[p] = \sum_{i=2}^{n} (\alpha_i - s_i \alpha_1)[a_i] + \sum_{j=2}^{m} (\beta_j - t_j \beta_1)[b_j].$$

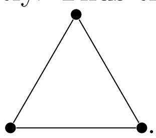
Let $x = \sum_{i=2}^{n} u_i[a_i] + \sum_{j=2}^{m} v_j[b_j]$. Comparing the coefficients with x, we see

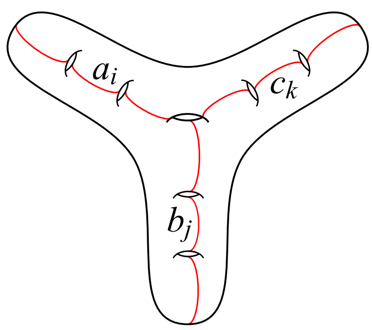
$$\begin{aligned} \alpha_i - s_i \alpha_1 &= u_i, \quad i = 2, \cdots, n, \\ \beta_j - t_j \beta_1 &= v_j, \quad j = 2, \cdots, m. \end{aligned}$$

Thus every coefficient is determined by α_1 and β_1. Since all the coefficients are non-negative, we get the constraints on α_1.

$$\begin{aligned} &\alpha_1 \ge 0, \\ &\alpha_1 \ge -u_i, \quad i = 2, \cdots, k, \\ &\alpha_1 \le u_i, \quad i = k+1, \cdots, n \end{aligned}$$

and similar constraints on β_1. This shows α_1 and β_1 take values in intervals independently. Thus the 2-cell is a rectangle.

Case 2: . The surface is homeomorphic to the picture below.



As before, the three families of curves are labeled $\{a_i\}_{i=1}^{m}$, $\{b_j\}_{j=1}^{n}$, $\{c_k\}_{k=1}^{l}$. The components provide relations

$$\sum_{i=1}^{n} s_i[a_i] + \sum_{j=1}^{m} t_j[b_j] = 0, \quad \sum_{i=1}^{n} s_i[a_i] + \sum_{k=1}^{l} r_k j[c_k] = 0,$$

where $s_i, t_j, r_k = \pm 1$. Next we show we can assume $s_1 = -1$ and $t_1 = r_1 = 1$ by relabeling or changing the signs of both equation simultaneously. If all the curves on each prong are pointing in the same way, i.e., $s_1 = \cdots = s_m$, $t_1 = \cdots = t_n$, $r_1 = \cdots = r_l$, the only possibility where C is reduced is $t_1 = r_1 = -s_1$. Thus we only need to adjust the sign. If there is a prong on which the curves point in different directions, we label that prong a. The only way we cannot arrange $t_1 = r_1 = 1$ is if $t_j = -r_k$ for all j, k, and that implies $b \cup c$ is trivial in homology, which is forbidden. By our choice of a, we can make $s_1 = -1$.

As before, let $p = \sum_{i=1}^{m} \alpha_i a_i + \sum_{j=1}^{m} \beta_j b_j + \sum_{k=1}^{l} \gamma_k c_k$ be a point in the 2-cell. Then

$$[p] = \sum_{i=1}^{m} (\alpha_i - s_i\beta_1 - s_i\gamma_1)[a_i] + \sum_{j=2}^{n} (\beta_j - t_j\beta_1)[b_j] + \sum_{k=2}^{l} (\gamma_k - r_k\gamma_1)[c_k].$$

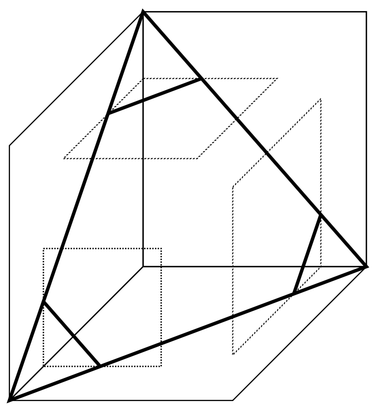
Let $x = \sum_{i=1}^{m} u_i[a_i] + \sum_{j=2}^{n} v_j[b_j] + \sum_{k=2}^{l} w_k[c_k]$. Comparing the coefficients with x, we see

$$\begin{aligned} \alpha_i - s_i\beta_1 - s_i\gamma_1 &= u_i, \quad i = 1, \cdots, m, \\ \beta_j - t_j\beta_1 &= v_j, \quad j = 2, \cdots, n. \\ \gamma_k - r_k\gamma_1 &= w_k, \quad k = 2, \cdots, l. \end{aligned}$$

When $i = 1$, we get $\alpha_1 + \beta_1 + \gamma_1 = u_1$. We can rewrite the first set of equations as

$$\alpha_i + s_i\alpha_1 = u_i + s_i u_1, \quad i = 2, \cdots, m.$$

Similar to Case 1, all coefficients are determined by $\alpha_1, \beta_1, \gamma_1$. The other equations may provide cutoff for $\alpha_1, \beta_1, \gamma_1$ individually. Thus we can only get n-gons up to $n = 6$.



With this information, it is not hard to construct examples to realize all of these possibilities. □