Torelli Groups

Math 8803
Spring 2018
Georgia Tech

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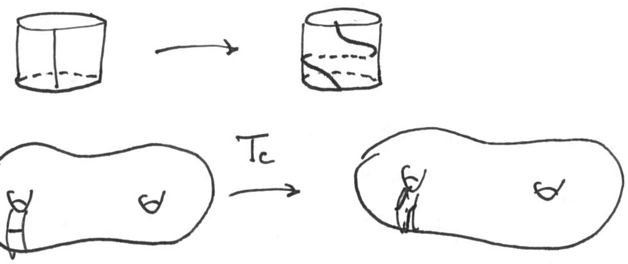
Background

Mapping class group:

S = surface.

$\mathrm{Mod}(S) = \pi_0 \, \mathrm{Homeo}^+(S, \partial S)$

Dehn twists:



<u>Thm</u> (Dehn 20's) For $g \geq 0$ $\mathrm{Mod}(S_g)$ is finitely generated by Dehn twists.

Symplectic rep $\quad \psi : \mathrm{Mod}(S_g) \longrightarrow Sp_{2g}(\mathbb{Z}) \quad$ action on $H_1(S_g; \mathbb{Z})$

<u>Thm</u>. For $g \geq 0$ ψ is surjective.

Torelli group: $\mathcal{I}(S_g) = \ker \psi$

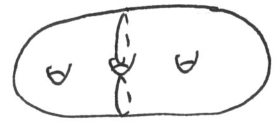
e.g. $T_c \in \mathcal{I}(S_g)$ for any sep curve c.

Why study Torelli?

1. It is the non-linear part of $\mathrm{Mod}(S_g)$
2. It is π_1 (Torelli space)
   ↑ space of Riem. surf's w/ H_1-basis.
3. Every $\mathbb{Z}HS^3$ obtained from S^3 by cutting along S_g, regluing by $\mathcal{I}(S_g)$

I. Generation

<u>Thm</u> (Johnson '83). For $g \geq 3$ $\mathcal{I}(S_g)$ is finitely gen by bounding pair maps



$T_c T_d^{-1}$

Putman '12: # gens cubic in g.

Thm (Mess '86) $\mathcal{I}(S_2) \cong F_\infty$.

II. Johnson Homomorphism

$$\tau : \mathcal{I}(S_g^1) \longrightarrow \Lambda^3 H \qquad H = H_1(S_g^1; \mathbb{Z}).$$

three defns: algebra, alg. top., 3-manifolds
application: $\mathcal{I}(S_g)$ is distorted in $\mathrm{Mod}(S_g)$ (Broaddus-Farb-Putman)

$\mathcal{K}(S_g) = \ker \tau$ "Johnson Kernel"

<u>Thm</u> (Johnson '83) $\mathcal{K}(S_g) = \langle T_c : c \text{ sep.} \rangle$

<u>Thm</u> (Ershov-He '17) $\mathcal{K}(S_g)$ is finitely gen.

Johnson filtration: $\underset{\mathcal{I}(S_g)}{\overset{}{\underset{\|}{N_0(S_g)}}} \trianglerighteq \underset{\mathcal{K}(S_g)}{\underset{\|}{N_1(S_g)}} \trianglerighteq \cdots$

III. The abelianization

Birman-Craggs-Johnson homomorphisms

$$\mathcal{I}(S_g) \longrightarrow \mathbb{Z}/2$$

defined using Rochin invariant for 3-manifolds.

There are $\sum_{k=0}^{3} \binom{2g}{k}$ of these.

<u>Thm</u> (Johnson '83) The abelianization of $\mathcal{I}(S_g)$ is given by

$$\tau \oplus BCJs$$

Also: Thm (Pitsch '08) Every $\mathbb{Z}HS^3$ obtained via $N_3(S_g)$.

IV. Higher finiteness properties

<u>Thm</u> (Johnson-Millson-Mess '83) $H_3(\mathcal{I}(S_3);\mathbb{Z})$ is ∞-gen.

<u>Thm</u> (Bestvina-Bux-M '08) $H_{3g-5}(\mathcal{I}(S_g);\mathbb{Z})$ is ∞-gen

Big Q. Is $H_2(\mathcal{I}(S_g);\mathbb{Z})$ finitely gen? Other H_k?

V. Representation stability

Johnson: parametrized Abel-Jacobi maps

$$\tau_i : H_i(\mathcal{I}_g^1; \mathbb{Q}) \longrightarrow \Lambda^{i+2} H \qquad 0 \le i \le 2g-2$$

<u>Thm</u> (Church-Farb '11)
- τ_i not injective $i > 1$
- τ_2 surjective
- τ_i nonzero $1 \le i \le g$ $(\Rightarrow H_i \neq 0)$

invented/conjectured
$\rightsquigarrow$ rep. stability

<u>Thm</u> (Boldsen-Dollerup '17) $H_2(\mathcal{I}(S_g); \mathbb{Z})$ finitely generated as an Sp-module.

<u>Also</u> Thm (Church-Putman '15) Fix k. Each $N_k(S_g)$ is generated by elements of small support, indep. of g.