Math 8803 Homework 2

Summary of Ingrid Irmer's paper[1]

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Consider the complex $C_h(S_g^1)$ for a primitive class $h \in H_1(S_g^1)$, where S_g^1 is the surface of genus g with 1 boundary component. The vertices are homotopy classes of oriented simple closed curves representing h. Two vertices u, v are connected if they bound a genus 1 subsurface F. If $\partial F = v - u$, then the direction of the edge points to v. Since $C_h(S_g^1)$ is connected, for any two vertices u, v, we can find a path connecting them. Define the signed distance $d_s(u, v)$ to be the number of edges traversed in the positive direction minus the number in the negative direction. We will show that d_s is well defined, that is, independent of the choice of path.

Define a map $\phi : \mathcal{I}(S_g^1) \times H_1(S_g^1) \to \mathbb{Z}$ by $\phi(\tau, h) = d_s(v, \tau v)$ where v is a vertex in $C_h(S_g^1)$. We will show in that ϕ does not depend on the choice of v. For a fixed τ, $\phi(\tau, \cdot)$ is a map from $H_1(S_g^1)$ to $\mathbb{Z}$. We will see that this is a homomorphism. Hence $\phi(\tau, \cdot)$ defines an element in $H^1(S_g^1)$. Thus we get a map $\phi : \mathcal{I}(S_g^1) \to H^1(S_g^1)$. The main theorem is

Theorem 1. *ϕ is equal to $\mathrm{Ch}/2$, where Ch is the Chillingworth class.*

1 Distance is well-defined

For two vertices u, v, fix a path $(v_0 = u, v_1, \cdots, v_k = v)$. For each edge (v_{i-1}, v_i) in the path, by definition there is a genus 1 subsurface F_i with $\partial F = v_i - v_{i-1}$. Also by definition, the orientation of F_i is the same as the surface S_g^1 if the edge points to v_i and vice versa. Let $H = F_1 + \cdots + F_k$. Then H is a 2-chain with boundary $v - u$. H is called the **trace** of the path.

Next we define the **preimage function**. For a 2-chain H and $x \in S_g^1 \setminus \partial H$, define $p_H(x) = [H] \frown [x] \in H_0(S_g^1) \cong \mathbb{Z}$, where $[H] \in H_2(S_g^1, \partial H)$, $[x] \in H^2(S_g^1, \partial H)$. Clearly, $p_H(x)$ is locally constant in $S_g^1 \setminus \partial H$.

To make the preimage function well behaved with respect to the addition of 2-chains, we need to define p_H on ∂H. We assume $\partial H = v - u$ where u and v are smooth simple closed curves that intersect transversely. This is to make sure that a small neighborhood of $x \in \partial H$ is cut into exactly 2 or 4 pieces by u and/or v. Then we define $p_H(x)$ to be the average of the function values of p_H on the pieces. With this definition, we have

Lemma 1. *If $H = \sum_i H_i$, then $\sum_i p_{H_i} = p_H$.*

Since $H_2(S_g^1) = 0$, If two 2-chains have the same boundary, then the difference is zero in homology. Therefore,

Lemma 2. *p_H only depends on ∂H.*

We introduce the integral with respect to Euler characteristics (see [2]). For a finite simplicial complex X and a function f which is constant on each open simplex, define

$$\int_X f d\chi = \sum_\sigma (-1)^{\dim \sigma} f(\sigma),$$

where the sum ranges over simplices σ. Note the integral of 1 is the Euler characteristic of X. More generally we define the geometric Euler characteristics for a (not necessarily closed) subcomplex A by the alternating sum of numbers of simplices of different dimensions. Then $\int_X f d\chi = \sum_c c\chi(f^{-1}(c))$ where the sum is over function values of f. The integral is clearly linear.

Now return to the signed distance. We have

Lemma 3. *The signed length of a path from u to v is $\frac{-1}{2}\int_X p_H d\chi$ for the trace H.*

Proof. Let $H = F_1 + \cdots + F_k$ be the trace, where the decomposition is as the construction in the beginning of this section. Since F_i is a subsurface of genus 1 with 2 boundary components, we can compute by definition to get $\int_X p_{F_i} d\chi = -2$ if the corresponding edge is traversed in the positive direction and $\int_X p_{F_i} d\chi = 2$ otherwise. The result follows from the linearity of the integral and Lemma 1. □

Corollary 1. *$d_s(u, v) = \frac{-1}{2}\int_X p_H d\chi$ for any H with $\partial H = v - u$. Thus d_s is well defined.*

Proof. This follows directly from Lemma 2 and Lemma 3. □

Since the boundary completely determines the preimage function, we will call any 2-chain H with $\partial H = v - u$ a trace from u to v.

Remark. Note we only defined $\phi(\tau, h)$ for primitive classes h. An obvious definition for all h is to factor h as kh_1, where h_1 is primitive, and define $\phi(\tau, h) = k\phi(\tau, h_1)$. A more natural way is to extend the definition of $C_h(S_g^1)$ to non-primitive classes. The vertices are homotopy classes of oriented multicurves with integral weights. Two vertices u, v are connected if v and u differ by the boundary of a genus 1 subsurface. The direction is defined by the orientation of the subsurface as before. The proof in the section only uses the correspondence between edges and genus 1 subsurfaces. Therefore, they work for non-primitive classes as well.

2 Signed stable length is a cohomology class

Recall $\phi(\tau, h) = d_s(v, \tau v)$ for some $v \in C_h(S_g^1)$. We have shown that $d_s(v, \tau v)$ does not depend on the path from v to τv. We still need to show that it does not depend on the choice of v to make ϕ well defined.

Lemma 4. *For a fixed $\tau \in \mathcal{I}(S_g^1)$, $h \mapsto \phi(\tau, h)$ is a well-defined homomorphism.*

Proof. Let $u, v \in C_h(S_g^1)$. Let H_v be the trace from v to τv. Since u and v are homologous, there is a 2-chain F_0 such that $\partial F = v - u$. Let $H_u = F + H_v - \tau F$. Then H_u is a 2-chain with $\partial H_u = \tau u - u$. Thus

$$d_s(u, \tau_u) = \frac{-1}{2} \int_X p_{H_u} d\chi = \frac{-1}{2} \int_X p_{H_v} d\chi = d_s(v, \tau v).$$

The integrals of p_F and $p_{\tau F}$ cancel because τ is an orientation preserving homeomorphism.

The same argument can be used to show the homomorphism property. Let $h_1, h_2 \in H_1(S_g^1)$. Let $u_1 \in C_{h_1}(S_g^1)$, $u_2 \in C_{h_2}(S_g^1)$, $v \in C_{h_1+h_2}(S_g^1)$. Let H_{u_1} and H_{u_2} be the corresponding traces. As before, there exists a 2-chain G with $\partial G = u_1 + u_2 - v$. Then $G + H_{u_1} + H_{u_2} - \tau G$ is a trace from v to τv. Taking the integral shows $d_s(u_1, \tau u_1) + d_s(u_2, \tau u_2) = d_s(v, \tau v)$. □

By identifying $H^1(S_g^1)$ with $\mathrm{Hom}(H_1(S_g^1), \mathbb{Z})$, we get a map $\phi : \mathcal{I}(S_g^1) \to H^1(S_g^1)$.

Lemma 5. *$\phi : \mathcal{I}(S_g^1) \to H^1(S_g^1)$ is a homomorphism.*

Proof. $\mathcal{I}(S_g^1)$ acts on $C_h(S_g^1)$ since it acts on curves and 2-chains. The action clearly preserves the length of paths. For $\tau, \sigma \in \mathcal{I}(S_g^1)$, $h \in H_1(S_g^1)$, $v \in C_h(S_g^1)$,

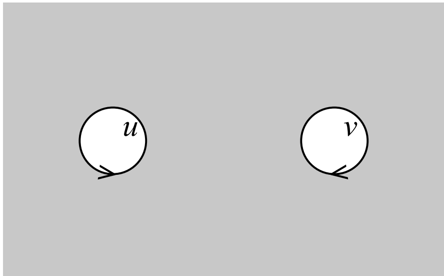
$$d_s(v, (\sigma\tau)v) = d_s(v, \sigma v) + d_s(\sigma v, \sigma(\tau v)) = d_s(v, \sigma v) + d_s(v, \tau v).$$

Thus $\phi(\sigma\tau, h) = \phi(\sigma, h) + \phi(\tau, h)$. □

Proof of the Theorem. Since $\mathcal{I}(S_g^1)$ is generated by bounding pair maps of genus 1, we only need to verify that they agree on these generators. Let $\tau = T_c T_d^{-1}$ where c and d bound a genus 1 subsurface. Choose a geometric symplectic basis $a_1 = c, b_1, \cdots, a_g, b_g$. Clearly τ fixes a_1 and $a_2, b_2, \cdots, a_g, b_g$. b_1 and τb_1 are disjoint simple closed curves that bound a subsurface F of genus 1, and $\partial F = \tau b_1 - b_1$. Thus $\phi(\tau, b_1) = 1$. This shows $\phi(\tau, \cdot) = \hat{i}(a_1, \cdot)$, which agrees with $\mathrm{Ch}(\tau)/2$. □

3 Topological proof

Recall that originally the Chillingworth class is defined using vector fields and winding numbers. Let X be a nonsingular vector field on S_g^1. Then $\mathrm{Ch}(\tau)([\gamma])$ for $\tau \in \mathcal{I}(S_g^1)$ and γ a simple closed curve is defined as $\omega_X(\tau\gamma) - \omega_X(\gamma)$ where



ω_X denotes the winding number of X relative to the tangent vector of the curve. Since γ and $\tau\gamma$ are connected in $C_{[\gamma]}(S_g^1)$, we can evaluate the difference in winding numbers one edge at a time. An edge $u \to v$ in $C_{[\gamma]}(S_g^1)$ is a genus 1 subsurface as in the figure. We cap off the boundary components and fill in the vector field such that on each radial lines the vectors are parallel and that the centers of the disks are zeros of the vector fields. The index of a zero is the winding number of X relative to a constant vector field along a small counterclockwise circle. Since the tangent vector makes a full counterclockwise turn, we see $\omega_X(u) = \text{index}(u) - 1$, $\omega_X(v) = -\omega_X(-v) = 1 - \text{index}(v)$. By the Poincaré-Hopf theorem, the sum of the indices is the Euler characteristics. Thus $\omega_X(v) - \omega_X(u) = 2 - (2 - 2g) = 2g = 2$, i.e., each edge contributes 2 to the winding number. Therefore, the signed distance is half the Chillingworth class.

References

[1] Irmer, Ingrid. "The Chillingworth class is a signed stable length." Algebraic & Geometric Topology 15.4 (2015): 1863-1876.

[2] Baryshnikov, Yuliy, and Robert Ghrist. "Target enumeration via Euler characteristic integrals." SIAM Journal on Applied Mathematics 70.3 (2009): 825-844.