Math 6421

Online or in person fine.
Some HW per week (drop 3) Gradescope.
Online office hours Fri 1:30-2:30 Teams 
& by appt.
Overview of Course

What is alg. geom?
study of solns of polynomials
(using ring theory, etc.)
“linear alg, without the linear”
much harder.
Setup

$k = \text{field}$ (usually can think $k = \mathbb{C}$)

$k[x_1, ..., x_n] = \text{ring of polynomials in } x_1, ..., x_n$ with coeffs in $k$.

$\mathbb{A}^n = \mathbb{A}^n_k = \text{affine n-space over } k$

$= \{(a_1, ..., a_n) \in \mathbb{A} \mid a_i \in k\}$

In bijection with $k^n$. In $\mathbb{A}^n$ no vect. sp. structure, so $O$ not special etc.
For \( f_1, \ldots, f_r \in k[x_1, \ldots, x_n] \):

\[
Z(f_1, \ldots, f_r) = \{(a_1, \ldots, a_n) \in \mathbb{A}^n : f_i(a_1, \ldots, a_n) = 0 \ \forall \ i\}
\]

\( Z \) \quad \text{zero set or vanishing set}

Some texts use \( V \) instead of \( Z \)

These are \underline{affine algebraic varieties}.

\underline{Special cases}

1. \( n = r = 1 \)
   - \( k \) alg closed: exactly \( d \) solns (with mult)

2. Linear Algebra
   - Solving polynomials in 1 var.
Again, much harder when \( \text{deg} > 1 \) or \( \# \text{eqns} > 1 \).

**Example**

\[
g_1(x, y) = 3x^3 - 17xy^2 + 2xy + 4y^2 - 6
\]

\[
g_2(x, y) = x^5 - x^3y^2 + 3xy^2
\]

We'll learn: \( \mathbb{Z}(g_i) \) is a "curve" in \( \mathbb{C}^2 \)

Generically: 0-dim soln set (points) \( \mathbb{Z}(g_2) \)
Weak Bezout's Thm

If $f_1, f_2 \in \mathbb{C}[x,y]$, no common factors
\[ \deg f_i = d_i \]

Then $|\mathbb{Z}(f_1, f_2)| \leq d_1 d_2$. 
Chapter 1  The geometry/algebra dictionary

\[ \text{Alg} \rightarrow \text{Geom} \]

Given \( J \subseteq k[x_1, \ldots, x_n] \) ideal

\[ \sim Z(J) = \{ a \in \mathbb{A}^n : f(a) = 0 \ \forall f \in J \} \]

example \( J = \langle f_1, \ldots, f_r \rangle \) ideal generated by \( f_1, \ldots, f_r \)

\[ = \{ g_1 f_1 + \cdots + g_r f_r : g_i \in k[x_1, \ldots, x_n] \} \]

Then \( Z(J) = Z(f_1, \ldots, f_r) \) as above.

\[ \text{Geom} \rightarrow \text{Alg} \]

Given \( V \subseteq \mathbb{A}^n \)

\[ \sim I(V) = \{ f \in k[x_1, \ldots, x_n] : f(a) = 0 \ \forall a \in V \} \]

this is an ideal.
We have:
\[
\{ \text{subsets of } \mathbb{A}^n \} \leftrightarrow \{ \text{ideals in } k[x_1, \ldots, x_n] \}
\]

Neither is injective. Why?

\[\iff \mathbb{Z}(x) = \mathbb{Z}(x^2) \]
more interesting direction

\[\implies \text{all open sets } \rightarrow \text{ O ideal in } \mathbb{C}.
\]

To fix latter, replace LHS with affine alg vars

For former, the example is the only issue. (taking powers).
The fix: For an ideal \( J \subseteq R \), have

\[
\text{rad}(J) = \{ r \in R : r^i \in J \text{ some } i \geq 1 \}
\]

"radical"

Will use Hilbert's Nullstellsatz to show

\[
\{ \text{affine alg vars in } \mathbb{A}^n \} \leftrightarrow \{ \text{radical ideals in } k[x_1, \ldots, x_n] \}
\]

Also:

\[
\{ \text{irreducible alg vars in } \mathbb{A}^n \} \leftrightarrow \{ \text{prime ideals ...} \}
\]

\[
\mathbb{A}^n = \{ \text{pts in } \mathbb{A}^n \} \leftrightarrow \{ \text{maximal ideals} \}
\]
Chapter 2 Projective Varieties.

\[ \mathbb{P}^n = \mathbb{P}_k^n = (k^{n+1} - 0) / k^* \]

where \( v \sim w \) if \( v = cw \) \( c \in k^* \).

= set of lines in \( k^{n+1} \)

Write equiv class of \((a_0, \ldots, a_n)\)

as \([a_0: a_1: \ldots: a_n]\)

In \( \mathbb{P}^2 \) these are all same!

In \( \mathbb{A}^2 \) have different conics

\( \bigcirc \bigcirc \bigcirc \bigcirc \)

We will study zero sets in \( \mathbb{P}^n \)....

because: more symmetry.
Can regard $\mathbb{P}^n$ as $\mathbb{A}^n \cup \mathbb{P}^{n-1}$.

$k^{n+1}.$

$a_n = 1 \approx \mathbb{A}^n$

$a_0, \ldots, a_n$

\[ \mathbb{A}^n = \{ [a_0 : \ldots : a_n] : a_n \neq 0 \} \]

\[ \mathbb{P}^{n-1} = \{ [a_0 : \ldots : a_{n-1} : a_n] : a_n = 0 \} \]
Projective varieties

$f(a_0, \ldots, a_n)$ is not well-defined on $\mathbb{P}^n$

e.g. $f(x, y) = x + y^2$

$f(-1, 1) = 0$
$f(-2, 2) = 2$

So can't say $f([-1:1]) = 0$.

But, if $f$ is homogeneous (all terms have same degree $d$)

then $f(cv) = c^d f(v)$.

So $f(v) = 0 \iff f(cv) = 0$

So: get zero sets in $\mathbb{P}^n$ for homog. poly's.
So for $f_1, \ldots, f_r$ homogeneous,

$$\mathbb{Z}(f_1, \ldots, f_r) = \{ a \in \mathbb{P}^n : f_i(a) = 0 \ \forall \ i \}$$

We'll see:

1. Affine varieties have projective closures

   $$V \rightarrow \mathcal{O}$$

2. Cone on a proj. var. is an aff. variety.

So, the theories are closely related.
Next time: Better Bezout

\[ Z(f_1) \] curves in \( \mathbb{P}^2 \) of deg \( d_1 \)

\[ Z(f_2) \quad & \quad f_1, f_2 \text{ no common factors.} \]

Then \[ |Z(f_1) \cap Z(f_2)| = d_1 d_2 \] (count with multiplicity).
Overview. From last time:

Chapter 1. The geometry/algebra dictionary

\( f_1, \ldots, f_r \in k[x_1, \ldots, x_n] \)

\( Z(f_1, \ldots, f_r) = \{ a \in A^n : f_i(a) = 0 \ \forall \ i \} \)

“affine algebraic variety”

There is a bijection

\( \{ aav's \ \text{in} \ A^n \} \leftrightarrow \{ \frac{\text{rad ideals in}}{k[x_1, \ldots, x_n]} \} \)

\( V \mapsto I(V) \)

\( Z(J) \leftrightarrow J \)

Chapter 2. Projective varieties

\( \mathbb{P}^n : (k^{n+1} - 0)/k^* \)

\([a_0, \ldots, a_n]\) written \([a_0 : \ldots : a_n]\)

\( \mathbb{P}^n = A^n \cup \mathbb{P}^{n-1} \)

homogeneous poly's in \( k[x_0, \ldots, x_n] \)

\( \mapsto \) projective alg. var's

These are always compact and tend to have more symmetry/
info (e.g. intersections at \( \infty \)).
Chapter 3 Classical constructions

1. Segre embedding

\[ \varphi_{m,n} : \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^{(m+1)(n+1)-1} \]

consequence: product of varieties is a variety

example: \( g_1(x,y) = 3x^3 - 17xy^2 \)
\( g_2(z,w) = z^5 - w^2z^3 \)

Does \( \mathbb{Z}(g_1,g_2) \) work?
No, get \( \mathbb{Z}(g_1) \times \mathbb{P}^1 \)

\( \mathbb{P}^1 \times \mathbb{Z}(g_2) \)
and more...

2. Veronese embedding

\[ \nu_d : \mathbb{P}^n \to \mathbb{P}^{(d+n)-1} \]

reduces the degree

For example: "Fermat curve"
\[ Z(x_0^3 + x_1^3 + x_2^3) \subseteq \mathbb{P}^2 \]
maps intersection of 9 quadrics in \( \mathbb{P}^9 \)

Application: Complements of varieties are varieties

e.g. \( \text{Poly}_2(\mathbb{C}) = \{ ax^2 + bx + c : b^2 - 4ac \neq 0 \} \)
\( \text{GL}_2(\mathbb{C}) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc \neq 0 \} \)
Grassmannian

Gr, n = \{ r - \text{dim planes thru 0 in } k^n \}

Note: \( \text{P}^n = \text{Gr}, n+1 \)

Gr, n important in topology:

"classifying space for n-dim vector bundles"

We'll show this a proj. var

Plücker embedding:

\[ \text{Gr}, n \to \text{P}(\Lambda^n k^n) \]

"parameter space of widgets is a widget"

Blow up

(Fixing, not destroying)

The map \( \mathbb{A}^2 \setminus 0 \to \text{P}^1 \)

does not extend to 0.

The blowup of \( \mathbb{A}^2 \) at 0:

\[ \{ (\mathbf{Q}, l) \in \mathbb{A}^2 \times \text{P}^1 : \mathbf{Q} \in l \} \]

Application: resolving singularities.

\[ x^3 = y^2 \]
Chapter 4. Dimension, degree, smoothness

“expected properties”

\[ \dim V = \text{(size of max chain of varieties at } p\text{)} - 1 \]

We'll show: behaves like \( \dim \) in lin alg.

\[ \text{codim } V_i = c_i \quad (V_i \text{ irreducible}) \]

\[ \text{codim } V_1 \cap V_2 = c_1 + c_2 \]

generically.

Degree \( V \subseteq \mathbb{A}^n \) or \( \mathbb{P}^n \) \( k \) alg. closed

\[ \dim V = k. \]

\[ \text{deg } V = \text{generic/expected } \# \text{ intersections} \]

with \( n-k \) plane

For \( V = Z(f) \) “hypersurface”

\[ \text{deg } V = \text{deg } f \]

Helps understand number of solns when \( \dim = 0 \).
Smoothness A Variety is smooth exactly when it is a manifold... can use manifold theory.

Chapter 5. Curves in the plane.

Setup $f \in \mathbb{C}[x_0, x_1, x_2]$ homog.

$C = \mathbb{Z}(f) \subseteq \mathbb{P}^2$

Picture: 

When $\deg f = 2$

$C$ is a "conic"

Thm. (Five pts determine a conic)

Given $p_1, \ldots, p_5 \in \mathbb{P}^2$ \exists conic passing thru all $p_i$ (generically unique)

Bézout's thm. $C_1 = \mathbb{Z}(f_1)$ \hspace{1cm} $\deg f_1 = d_1$

$C_2 = \mathbb{Z}(f_2)$

& $f_1, f_2$ no common factor

Then $|\mathbb{Z}(f_1) \cap \mathbb{Z}(f_2)| = d_1d_2$

(count with mult.)

Thm. Given $(d+1)$ distinct pts in $\mathbb{P}^2$

\exists $\deg d$ curve passing thru.
Cayley-Bacharach Thm
$C_1, C_2 \subseteq \mathbb{P}^2$ cubic curves
with $|C_1 \cap C_2| = 9$
If $C_3$ passes thru 8 of the pts,
it passes thru the 9th.

Thm. \( f \in \mathbb{C}[x_0, x_1, x_2] \)
irred, homog, deg d
\( \sim \mathbb{Z}(f) \)
Then # sing pts \( \leq \binom{d-1}{2} \)

Classification of cubic curves in $\mathbb{P}^2$

Smoothness for curves:
\[ C = \mathbb{Z}(f) \]
smooth a p if some
\[ \frac{df}{dx_i}(p) \] nonzero.

Actual pic

\[ Z(x^2+y^3) \subseteq \mathbb{A}^2 \]
\[ Z(y^2 = x^2 + x^3) \subseteq \mathbb{A}^2 \]
"Weierstrass curves"
Chapter 6. Special topics

Cayley-Salmon thm

Every smooth cubic surface in $\mathbb{P}^3$ contains 27 lines.

e.g. $x^3 + y^3 + z^3 + w^3$

find the lines!
Chapter 1. Affine alg. vars

& the geometry/alg dictionary

Setup: \( S \subseteq k[x_1, \ldots, x_n] \)

\[ \mathcal{Z}(S) = \{ a \in \mathbb{A}^n : f(a) = 0 \text{ } \forall f \in S \} \]

“affine alg var”

Examples

\( \phi = \mathcal{Z}(k[x_1, \ldots, x_n]) = \mathcal{Z}(1) \)

Second = makes sense since \( (1) = k[x_1, \ldots, x_n] \)

\( \mathbb{A}^n = \mathbb{Z}(0) \)

\( (a_1, \ldots, a_n) = \mathbb{Z}(x_1-a_1, \ldots, x_n-a_n) \)

compare: lin alg.

\( \text{(Hyper)planes} \)

\( \text{Conics} \quad \mathbb{Z}(f) \subseteq \mathbb{A}^2 \quad \deg f = 2. \)

e.g. \( x^2-y^2-1 \quad xy-1 \quad y-x^2 \)

\[ (x-y)(x+y) \quad x^2-1 \quad x^2 \]

“double line”
Aside: Conics over \( \mathbb{C} \) from a topological pt of view.

1. \( \mathbb{Z}(xy-1) \) is connected over \( \mathbb{C} \), every pt connected to \((1,1)\).

2. \( \mathbb{Z}(x^2-y) \) over \( \mathbb{C} \).

Have a map

\[
\mathbb{Z}(x^2-y) \rightarrow \mathbb{C} \quad (x,y) \mapsto y
\]
Algebraic groups

\[ \text{SL}_{n,k} = \mathbb{Z} (\det -1) \leq \mathbb{A}^{n^2} \]

\[ \text{GL}_{n,k} \text{ complement of } \mathbb{Z} (\det) \]

by defn.

In general, complements of aav’s are aav’s (later)

To see \( \text{GL}_{n,k} \) as a variety:

\[ V = \left\{ (x_{ij}, t) \in \mathbb{A}^{n^2+1} : \det(x_{ij}) t - 1 = 0 \right\} \]

\[ \varphi: \text{GL}_{n,k} \to V \]

\[ A = (a_{ij}) \mapsto (a_{ij}, \frac{1}{\det A}) \]

is a bijection.

Twisted cubic

\[ C = \text{Im } \varphi \text{ where } \]

\[ \varphi: \mathbb{A}^1 \to \mathbb{A}^3 \]

\[ t \mapsto (t, t^2, t^3) \]

As a variety

\[ C = \mathbb{Z} (x^2 - y, x^3 - z) \]

\[ = \mathbb{Z} (x^2 - y, z - x y) \]

intersection of two “quadrics”

\[ C \text{ is also a determinantal var } \]

\[ C = \{ (x, y, z) \in \mathbb{A}^3 : \text{rank} \left( \begin{array}{ccc} x & y & z \\ x & y & z \\ x & y & z \end{array} \right) < 2 \} \]

(Chris)

Q. Is any int. of quadrics a det. var?
8. A family of (smooth) cubics

\[ C_{\lambda} = \mathbb{Z}(x(x-1)(x-\lambda) - y^2) \subseteq \mathbb{A}^2 \]
\[ \lambda \neq 0, 1 \quad k = \mathbb{C} \]

Claim: \( C_{\lambda} \cong \mathbb{C} \)

Like the \( x-y^2 \) example:

\[ C_{\lambda} \to \mathbb{A}^1 \]
\[ (x, y) \to x \]

Other than \( x = 0, 1, \lambda \)

pts in \( \mathbb{A}^1 \) have two preims.

9. Trefoil

\[ \mathbb{Z}(x^2 + y^2)^2 + 3x^2y - y^3) \]

intersect with \( S^3 = \{(x, y) : |x|^2 + |y|^2 = 1\} \)

"singularity theory"

exercise: Take complement of axes in \( \mathbb{C}^2 \)

& intersect with \( S^3 \).
**Nonexamples** \( k = \mathbb{C} \) in \( \mathbb{A}^n \)

1. **Fact.** Every affine algebraic variety is closed in Euclidean topology.
   \[ \mathbb{Z}(S) \text{ is not an a.a.v. in } \mathbb{A}^n \]

2. **Fact.** The interior of any algebraic variety is \( \emptyset \).
   \[ \mathbb{Z}(\{ z : |z| \leq 1 \}) \text{ is not an a.a.v.} \]

3. **Fact.** Any proper algebraic variety in \( \mathbb{A}^1 \) is finite (by FTAAlg).
   \[ \mathbb{Z}(z) \subset \mathbb{C} \text{ is not a.a.v.} \]

**Basic Properties of a.a.v's**

1. \( \forall S \subseteq k[x_1, \ldots, x_n] \) have
   \[ \mathbb{Z}(S) \cap \mathbb{Z}(S') = \mathbb{Z}(S \cup S') \]
   (exercise)

2. **Intersections of a.a.v's are a.a.v**
   \[ \bigcap_{\alpha} \mathbb{Z}(I_\alpha) = \mathbb{Z}(\bigcup_{\alpha} I_\alpha) \]

3. **Finite unions of a.a.v's are a.a.v**
   \[ V(I) \cup V(J) = V(IJ) \]
   (exercise)
   \[ \bigcup_{k,j} \mathbb{Z}(x_k x_j) \]
   \[ \bigcup_{k,j} \mathbb{Z}(x_k x_j) \]
   example \( V(x) \cup V(y) = V(xy) \)
Zariski Topology

A topology on a space $X$ is a collection of sets, called closed sets such that:

1. $\emptyset$, $X$ closed
2. Finite unions of closed sets are closed
3. Arbitrary intersections of closed sets are closed.

Complements of closed sets called "open".

Def. Zariski topology on $\mathbb{A}^n$ has aar's as the closed sets.

Basic properties $\Rightarrow$ this indeed is a topology.

The Zariski top. is strange:

1. All proper closed sets have $\emptyset$ interior.
2. Proper closed subsets of $\mathbb{A}^1$ are finite.
3. No two open sets are disjoint.
   $\Rightarrow$ not Hausdorff.
4. Compact $\Rightarrow$ closed
   closed $\Rightarrow$ compact

A concept: A set is Zariski dense iff every polynomial is det. by its values on that set.

E.g. $\mathbb{Z} \subseteq \mathbb{A}_R^1$
HW due Mon
Hilbert Basis Thm

Thm. Every \( \mathbb{Z}(I) \) equals some \( \mathbb{Z}(f_1, \ldots, f_r) \)
i.e. every \( a \) is the intersection of finitely many hypersurfaces

Lemma/Defn. \( R \) ring TFAE

1. Every ideal in \( R \) is fin gen.
2. \( R \) satisfies asc. chain cond:
   any \( I_1 \subseteq I_2 \subseteq \cdots \) eventually stationary.

Say \( R \) is Noetherian.

Fact. Fields are Noetherian.

PF of Lemma.

1 \( \implies \) 2. Let \( I_1 \subseteq I_2 \subseteq \cdots \)
   \( \implies I = U I_i \) is an ideal.

\( I \) is f.g. by 1.

Some \( I_j \) contains all gens

so \( I_k = I_j \) \( k \geq j \).

2 \( \implies \) 1. If \( I \) not f.g.

make \( I_1 \neq I_2 \neq I_3 \neq \cdots \)

by adding on gen. at a time.
Prop. $R$ Noetherian $\Rightarrow$ 
$R[x_1,\ldots,x_n]$ Noeth.

In our case $R = K$, so HBT follows.

If. We'll do $R[x]$, rest is induction.

Say $I \subseteq R[x]$ not f.g.
Let $f_0 = \text{non-zero elt of } I$ of min deg.

Given $f_i$: $f_{i+1} = \text{nonzero elt of } I \setminus \langle f_0,\ldots,f_i \rangle$ of min deg.

Note $\deg f_i \leq \deg f_{i+1}$

Let $a_i = \text{lead coeff of } f_i$.

$I_i = (a_0,\ldots,a_i) \subseteq R$.

$R$ Noeth $\Rightarrow I_0 \subseteq I_1 \subseteq \ldots$ eventually stat.

So if $m$ st $a_{m+1} \in (a_0,\ldots,a_m)$

$\Rightarrow a_{m+1} = \sum r_i a_i \quad r_i \in R$

Let $f = f_{m+1} - \sum_{i=0}^{m} x^{\deg f_{m+1} - \deg f_i} r_i f_i$

This $f$ cooked up so $\deg f < \deg f_{m+1}$

Thus $f \in J_m \Rightarrow f_{m+1} \in J_m$

contrad. \qed
Hilbert's Nullstellensatz c. 1900

Weak Nullst. $k$ alg closed

Every maximal ideal in $k[x_1, \ldots, x_n]$ is of form $(x_1 - a_1, \ldots, x_n - a_n)$.

Strong Nullst. $k$ alg closed

$I$ is $k[x_1, \ldots, x_n]$ ideal. Then

$$I(\overline{\text{Z}(I)}) = \sqrt{I}$$

i.e.

$$\{\text{assocs in } k[x_1, \ldots, x_n]\} \overset{\text{bij}}{\leftrightarrow} \{\text{rad. ideals in } k[x_1, \ldots, x_n]\}$$

$$X \mapsto I(X)$$

$$\overline{\text{Z}(I)} \leftrightarrow I$$

The WN implies other natural statements:

- Every proper ideal in $k[x_1, \ldots, x_n]$ has a common zero, i.e. $I \subseteq k[x_1, \ldots, x_n] \Rightarrow \overline{\text{Z}(I)} \neq \emptyset$
- Converse: a family of polynomials with no common zeros generates whole $k[x_1, \ldots, x_n]$. 
Aside: SN is a generalization of Fund Thm Alg.

First, note

$(f) \in \mathbb{C}[z]$ radical

$\iff f$ has no rep. roots.

SN $\Rightarrow$ FTA because

$I(\mathbb{Z}(f)) = \sqrt{(f)}$ implies

$f$ has a root.

FTA $\Rightarrow$ SN because

$f$ factors into linear.

$\Rightarrow I(\mathbb{Z}(f)) = \sqrt{(f)}$

PT by example:

$f(z) = (z-1)(z-3)^2$

$I(\mathbb{Z}(f)) = I(\{1,3\}) = (z-1)(z-3)$

$= \sqrt{(f)}$. 
Both WN & SN fail for $k$ not alg. closed:

\[ (x^2 + 1) \text{ radical in } \mathbb{R}[x] \]

since $\mathbb{R}[x]/(x^2 + 1) \cong \mathbb{C}$

But $\mathbb{I}(\mathbb{Z}(x^2 + 1)) = \mathbb{I}(\emptyset) = \mathbb{R}[x]$. 
Proof of $\text{WN} \Rightarrow \text{SN}$

"Trick of Rabinowitz"

Say $g \in I(Z(f_1, \ldots, f_m))$

Want $g$ some power $c_{(f_1, \ldots, f_m)}$.

The assumption $\Rightarrow$ a common zero of the $f_i$ is a zero of $g$.

Thus $f_1, \ldots, f_m, x_{n+1}g - 1$ have no common zeros in $\mathbb{A}^{n+1}$.

$\text{WN} \Rightarrow (f_1, \ldots, f_m, x_{n+1}g - 1) = k[x_1, \ldots, x_{n+1}]$

$\Rightarrow 1 = p_1f_1 + \cdots + pmf_m + p_{m+1}(x_{n+1}g - 1)$

where $p_i \in k[x_1, \ldots, x_{n+1}]$

Apply the map

$k[x_1, \ldots, x_{n+1}] \rightarrow k(x_1, \ldots, x_n)$

$x_i \mapsto x_i$

$x_{n+1} \mapsto \frac{1}{g}$

$\Rightarrow 1 = p_i(x_1, \ldots, x_n, \frac{1}{g})f_i + \cdots + p_m(x_1, \ldots, x_n, \frac{1}{g})f_m$ in

$= \frac{\text{something in } (f_1, \ldots, f_m)}{g \text{ power}}$
Fact. Each \((x_1-a_1, \ldots, x_n-a_n)\) is maximal.

If \(k[x_1, \ldots, x_n] / (x_1-a_1, \ldots, x_n-a_n) \rightarrow k\)

\[
\begin{align*}
f & \quad \rightarrow \quad f(a_1, \ldots, a_n) \\
1 & \quad \leftrightarrow \quad 1 \\
\end{align*}
\]

This is so done.
Thm. \( k \) = field, \( K \) extension

If \( K \) is fin gen as a \( k \)-alg
then \( K \) is algebraic over \( k \).

Pf of WN. Say \( m = \text{max ideal in} \)

\[ R = k[x_1, \ldots, x_n] \]

\[ \implies R/m \text{ is a field, fin gen as } k\text{-alg.} \]
(since \( R \) is).

Have \( k \cap m = \{0\} \). (else \( m = R \))

\[ \implies \text{image } \bar{k} \text{ of } k \text{ in } R/m \text{ is } = k. \]

Thm \( \implies R/m \text{ alg. ext. of } \bar{k}. \)

\( k \text{ alg closed } \implies R/m = \bar{k} \)

Under \( R \to R/m \)
each \( x_i \mapsto \bar{a}_i \in \bar{k} \)
some \( \bar{a}_i \) image of \( a_i \in k \).

\[ \implies m = (x_i-a_i, \ldots, x_n-an) \]

But \( m \) maximal

\[ \implies m = m' \]

Hilbert's N'satz

\{ aa's in M^n \} \leftrightarrow \{ \text{rad ideals in } k[x_1, \ldots, x_n] \}

V \mapsto \Pi(V)
Z(V) \leftrightarrow I

Nontrivial part: \( Z \) inj on rad ideals

Weak N'satz Max ideals in \( K[x_1, \ldots, x_n] \) are of form
\( (x_1 - a_1, \ldots, x_n - a_n) \)

Lemma. Assume \( k \) alg closed and uncountably infinite.

If \( L \supseteq k \) field ext and \( L \) fin. gen. as \( k \)-algebra

Then \( L \) is algebraic over \( k \).

\( \exists u_1, \ldots, u_r \in L \) so that each elt of \( L \) is a polynomial in \( u_i \) with coeffs in \( k \).

Example \( C(x) \) not alg over \( C \), not fg alg
Lemma. Assume \( k \)-alg closed and uncountably infinite.

If \( L \supseteq k \) field ext. and \( L \) fin. gen. as \( k \)-algebra
Then \( L \) is algebraic over \( k \).

Pf. Suppose \( u \in L \) not algebraic.

1. The set \( \{ u^{-c} : c \in k \} \) uncountable and lin. ind. over \( k \).
   Indeed, any lin. combo
   \[ \frac{b_1}{u-c_1} + \ldots + \frac{b_q}{u-c_q} \]
   gives \( u \) as a root of a poly (clear fractions)

2. Let \( u_1, \ldots, u_r \) gens for \( L \) as \( k \)-alg.
   \( \{ u_1, u_2, \ldots, u_r \} \) countable and is a \( k \)-basis for \( L \).
   This contradicts 1.

Lemma is true over arbitrary fields. Need
1. Zariski's Lemma: \( L \) fin. gen. as \( k \)-alg \( \iff L \) fin. gen. as \( k \)-module
2. Noether normalization
Lemma. \( k = \text{field}, \ K = \text{extension} \)

If \( K \) is fin gen as a \( k \)-alg

then \( K \) is algebraic over \( k \).

Proof of WN. Say \( m = \text{max ideal in} \)

\[ R = k[x_1, \ldots, x_n] \]

\( \Rightarrow R/m \) is a field, fin gen as \( k \)-alg.

(since \( R \) is).

Have \( k \cap m = \{ 0 \} \). (else \( m = R \))

\( \Rightarrow \) image \( \overline{k} \) of \( k \) in \( R/m \) is \( \cong k \).

Lemma.

\( \Rightarrow R/m \) alg. ext. of \( \overline{k} \).

\( k \) \( \text{alg closed} \Rightarrow R/m = \overline{k} \)

Under \( R \to R/m \)

each \( x_i \mapsto \overline{a}_i \in \overline{k} \)

some \( \overline{a}_i \) image of \( a_i \in k \).

\( \Rightarrow m \cong (x_1-a_1, \ldots, x_n-a_n) \)

\( m' \)

But \( m' \) maximal

\( \Rightarrow m = m' \) \( \square \)
Irreducibility

Basic example

1. $\mathbb{Z}(xy) \subseteq \mathbb{A}^2$
   $\mathbb{Z}(x) \cup \mathbb{Z}(y)$

Say $\mathbb{Z}(xy)$ reducible.

An aav is reducible if it is the union of two distinct, nonempty aav's.

The maximal irreducible closed subsets are the irreducible components.

More examples

2. $\mathbb{Z}(x_1 x_2, x_1 x_3) \subseteq \mathbb{A}^3$
   $\mathbb{Z}(x_1) \cup \mathbb{Z}(x_2 x_3)$

3. $\mathbb{Z}(x^2 - 1) \subseteq \mathbb{A}^1$
   $\mathbb{Z}(x+1) \cup \mathbb{Z}(x-1)$

4. A finite set in $\mathbb{A}^n$ is irreducible if it is not connected.

5. What about $\mathbb{A}^n$?
Prop. \( X \subseteq \mathbb{A}^n \) aav.

\( X \) irred \( \iff \) \( \mathcal{I}(X) \) prime.

\[ \begin{align*}
\text{IF} & \quad \iff \text{Say } \mathcal{I}(X) \text{ prime.} \\
\text{and } X &= X_1 \cup X_2 \\
\text{Then } \mathcal{I}(X) &= \mathcal{I}(X_1) \cap \mathcal{I}(X_2) \\
\mathcal{I}(X) \text{ prime } &\Rightarrow \mathcal{I}(X) = \mathcal{I}(X_1) \text{ wlog}
\end{align*} \]

(If \( P = I \cap J \) then \( I \cap J \subseteq I \cap J = P \Rightarrow P = I \lor J \).

\( \text{(Prime } \Rightarrow \text{ radical) so } SN \Rightarrow \) 

\( X = X_1 \).

\(<\Rightarrow> \) Say \( X \) irred & \( fg \in \mathbb{I}(X) \)

Then \( X \subseteq Z(fg) = Z(f) \cup Z(g) \)

\[ \begin{align*}
\Rightarrow X &= (Z(f) \cap X) \cup (Z(g) \cap X) \\
\text{irred.} &\Rightarrow X = Z(f) \cap X \\
\Rightarrow X \subseteq Z(f) &\Rightarrow f \in \mathbb{I}(X). \quad \square
\end{align*} \]
Consequences

1. \( \mathbb{A}^n \) irreducible since (0) prime
2. \( f \in k[x_1, \ldots, x_n] \) irreducible
   \[ \iff Z(f) \text{ irreducible.} \]

\[
\begin{bmatrix}
\text{If } f = f_1 f_2 \\
Z(f) = Z(f_1) \cup Z(f_2)
\end{bmatrix}
\]

Dictionary

- aav's \( \leftrightarrow \) rad ideals
- irreducible aav's \( \leftrightarrow \) prime ideals
- (in \( A^n \)) pts \( \leftrightarrow \) max ideals
  (in \( k[x_1, \ldots, x_n] \))

Decomposing into irreducibles

- \( k[x_1, \ldots, x_n] \) Noetherian (Hilb. basis thm)
  \[ \rightarrow \text{any desc. chain of aav's is eventually stationary.} \]
  (Noetherian property for aav's)

Prop.1: An aav can be written as a finite union of irreducible aav's

- \( X_1, \ldots, X_r \)

Prop.2: If \( X_i \cap X_j \forall i \neq j \) the \( X_i \) unique.

In this case, \( X_i \) called the irreducible components of \( X \).
Prop. 1: An aav can be written as a finite union of irreducible aav's:
\[ X = X_1 \cup \cdots \cup X_r \]

2. If \( X_i \neq X_j \) \( \forall i \neq j \), the \( X_i \) are unique. In this case, \( X_i \) called the irreducible components of \( X \).

Proof of 1: Let \( X \) be a minimal counterexample. If there is a counterex, a minimal one exists by Noetherian property.

Since it's a counterex, it's reducible:
\[ X = X_1 \cup \cdots \cup X_r \]

But \( X \) minimal \( \Rightarrow \)
\[ X_1, X_2 \] finite unions of irreducible's.

2. Say
\[ X = X_1 \cup \cdots \cup X_r \]
\[ = X'_1 \cup \cdots \cup X'_s \]
\[ X_1 \subseteq U X'_i \]

In fact \( X_i \subseteq X_i \), some i.
(Otherwise \( X_i \) reduces.)
Next week: end of Chap 1

- Morphisms = polynomial maps

- Coordinate ring $k[V]$
  \[ = \{ \text{poly fns on } V \} \]
  \[ = k[x_1, \ldots, x_n] / \mathfrak{I}(V) \]
**Morphisms**

\( x \in \mathbb{A}^n, \, y \in \mathbb{A}^m \) aav's

\( f: X \rightarrow Y \) is a **morphism** if it's restriction of a polynomial map \( \mathbb{A}^n \rightarrow \mathbb{A}^m \).

i.e. \( \exists \, f_1, \ldots, f_m \in k[x_1, \ldots, x_n] \)

s.t. \( f(x) = (f_1(x), \ldots, f_m(x)) \quad \forall \, x \in X \).

A **morphism** is an **isomorphism** if it has an inverse morphism.

**Examples**

1. **Affine change of coords** \( \mathbb{A}^n \rightarrow \mathbb{A}^n \)
   
   Linear map + translation.
   
   This is \( \cong \iff \) linear map is.

2. \( C = \mathbb{Z}(\ y - x^2) \subseteq \mathbb{A}^2 \)

   \( f: \mathbb{A} \rightarrow C \)

   \( t \mapsto (t, t^2) \)

   \( f^{-1}: C \rightarrow \mathbb{A}^1 \)

   \( (x, y) \mapsto x \) isomorphism.

In general, coord fn are morphisms.
Facts about morphisms

1. Morphisms are continuous wrt Zariski topology
   \[ f: X \to Y \]
   \[ f^{-1}(\mathbb{Z}(h_1, \ldots, h_r)) = \mathbb{Z}(h_1 \circ f, \ldots, h_r \circ f) \]

2. Morphisms do not always map aav's to aav's
   \[ \mathbb{Z}(xy-1) \to \mathbb{A}^1 \]
   \[ (x, y) \mapsto x \]
   Image is \[ \mathbb{A}^1 \setminus 0 \]

3. \[ C = \mathbb{Z}(x^3 + y^2 - x^2) \]
   \[ f: \mathbb{A}^1 \to C \]
   \[ t \mapsto (t^2 - 1, t(t^2 - 1)) \]
   morphism, but not injective:
   \[ f(1) = f(-1) \]

4. \[ C = \mathbb{Z}(y^2 - x^3) \subseteq \mathbb{A}^2 \]
   \[ f: \mathbb{A}^1 \to C \]
   \[ t \mapsto (t^2, t^3) \]
   bijective morphism,
   but not \( \cong \). Why? We need a new tool...
Coordinate Rings

\[ X = \text{aaa} \]

\[ \sim k[X] = \{ f | f : k[x_1, \ldots, x_n] \} \]
\[ = \{ \text{poly \ fins \ on \ } X \} \]
\[ = \text{coord \ ring \ on \ } X. \]

\[ k[X] \text{ is a ring, in fact a } k-\text{algebra}. \]

More:

\[ k[X] = k[x_1, \ldots, x_n] / \mathfrak{I}(X) \]

So, if \( X = \mathbb{Z}(xy - 1) \)

\[ \{ y \} \in k[X] \quad (!) \]

First examples

1. \( k[ \mathbb{A}^n ] \cong k[x_1, \ldots, x_n] \)

2. \( k[\mathfrak{p}] \cong k \) (cf proof \( (x_1 - a_1, \ldots, x_n - a_n) \text{ maximal} \))

3. \( k[X] \cong k^r \)

\[ X = P_1 u \ldots u P_r \]

\[ f \mapsto (f(p_1), \ldots, f(p_r)) \]
More examples

4) \( L = \mathbb{Z}(y-mx-b) \)

\[ k[L] \cong k[x] \]

First, any poly in \( x,y \)

is equiv to a poly in \( x \)

\[ y \sim mx+b \]

\[ k[L] \rightarrow k[x] \]

\[ [f(x,y)] \rightarrow f(x, mx+b) \]

\[ k[x] \rightarrow k[L] \]

\[ f(x) \mapsto [f(x)] \]

These are inverses.

5) \( C = \mathbb{Z}(x^2+y^2-z^2) \subseteq \mathbb{A}^3 \) cone

\( k = C \)

exercise: in \( k[C] \)

\((x^3 + 2xy^2 - 2xz^2 + x) \sim (x - x^3)\)
Irreducibility & Coord rings

Prop. $X$ irreducible $\iff k[X]$ integral domain

$\text{PF. } X$ irreducible $\iff \Pi(X)$ prime

$\iff k[x_1,\ldots,x_n]/\Pi(X)$ integral domain

Fact. $k[X]$ gen. by coord fns,

$X \to k$

$(a_1,\ldots,a_n) \mapsto a_i$

hence the name (?)

Twisted cubic

$C = \mathbb{Z}(y-x^2, z-x^3)$

Will show $C$ is irreducible.

$\text{PF #1 } (y-x^2, z-x^3)$ prime.

Use $k[x,y,z] = (k[x,y])[z]$

$k[x,y] = (k[x])[y]$

Suppose $f,g \in (y-x^2, z-x^3)$.

Division alg:

$f(x,y,z) = (z-x^3)f_1(x,y,z) + \text{remainder (const in z)}$

$(z-x^3)f_1(x,y,z) + (y-x^2)f_2(x,y) + f_3(x)$

similar $g(x,y,z) = (z-x^3)g_1(x,y,z) - (y-x^2)g_2 + g_3(x)$

Since $f,g \in (y-x^2, z-x^3)$ $\Rightarrow f_3 = 0$ or $g_3 = 0$
Indeed if \( f_3(x) \) & \( g_3(x) \) both nonzero, can find a point in \( \mathbb{A}^3 \) where \( y-x^2, z-x^3 \) vanish but \( fg \) does not.

Back to the dictionary

\[
\text{sub-aaav's of } X \leftrightarrow \text{rad. ideals in } k[X] \\
y \subseteq X \rightarrow k[Y] \subseteq k[X]
\]

irred \( \leftrightarrow \) prime
pts \( \leftrightarrow \) max ideals.

**Pf.** 3rd \( \equiv \) thm + prev. dictionary

Next time Every fin gen., reduced \( k\)-alg is some \( k[X] \).
Last time

1. Morphisms $X \to Y$
   - polynomial map

2. Coordinate rings
   - $k[X]$: poly. fns on $X$
     - $\{ f|_X : f \in k[x_1, \ldots, x_n] \}$
   - $k[x_1, \ldots, x_n] / \Pi(X)$

Wanted to show:

$\mathbb{A}^1 \to \mathbb{Z}(y^2 - x^3)$ not $\cong$.

Next: A morphism $X \to Y$ gives hom. $k[Y] \to k[X]$

Pullbacks. $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ and:

$f: X \to Y$ morphism.

$\sim \to f_* : k[Y] \to k[X]$

$g + \Pi(Y) \mapsto g \circ f + \Pi(X)$

or $[g] \mapsto [g \circ f]$

Basic facts:

1. $f_*$ is $k$-alg homom.
2. $(fg)_* = g_* f_*$
3. $f^* an \cong \Rightarrow f_* an \cong$
Basic facts:
1. $f^*_*$ is $k$-alg homom.
2. $(fg)^*_* = g^*_* f^*_*$
3. $f^* a_n \cong \Rightarrow f^*_* a_n \cong$

**Contravariant**

In other words, have a functor:

\[ \text{aav's} \rightarrow k\text{-algebras} \]

\[ X \rightarrow k[[X]] \]

What is the image?

---

Examples:
1. $\mathbb{A}^1 \cong \mathbb{Z}(y-x^2) \subseteq \mathbb{A}^2$
   \[ t \mapsto (t, t^2) \]

Pullback:
\[ f: C[x,y]/(y-x^2) \rightarrow C[t] \]
\[ g_1(x,y) = x \rightarrow t \]
\[ g_2(x,y) = y \rightarrow t^2 \]

This is enough since $x, y$ generate.

Surjective \checkmark

Injective \checkmark

So $\cong$
2. \( f: A' \rightarrow \mathbb{Z}(y^2 - x^3) \subseteq A^2 \)

\[
t \mapsto (t^2, t^3)
\]

Would be better:

\[
\mathbb{C}[x, y]/(y^2 - x^3) \nleq \mathbb{C}[t]
\]

Joshua's idea: compare transcendence degree over \( \mathbb{C}[x] \).

(Can a transc. ext. of \( \mathbb{C}[x] \) be \( \approx \) to \( \mathbb{C}[x] \)?)

More refined: LHS free module over \( \mathbb{C}[t] \) of rank 2 and RHS rank 1.

Pullback:

\[
\mathbb{C}[x, y]/(y^2 - x^3) \rightarrow \mathbb{C}[t]
\]

\[
x \mapsto t^2
\]

\[
y \mapsto t^3
\]

Not surj: \( t \) not in image

so \( f \) is not an \( \approx \).
Suppose $\Phi : k[x, x^{-1}] \to k[x]$

$\Rightarrow \Phi(1) = 1$
$\Rightarrow \Phi(x) \Phi(x^{-1}) = 1$
$\Rightarrow \Phi(x), \Phi(x^{-1})$ units

$\Rightarrow \operatorname{Im} \Phi \subseteq \{ \text{constant polys} \}$

Next: Which alg's arise?
Defn. An alg is reduced if no nilpotent elts, i.e. no elts \( r \neq 0 \) with \( r^k = 0 \).

Thm
1a. Every \( k[X] \) is a fin gen. reduced \( k \)-alg.
1b. Every fin gen red. \( k \)-alg is a \( k[X] \).
2a. \( f: X \rightarrow Y \) morphism \( \Rightarrow f_*: k[Y] \rightarrow k[X] \) \( k \)-alg homom.

2b. Every \( k \)-alg homom \( R \rightarrow S \) of red fin gen \( k \)-alg is some \( f \) & \( f \) unique up to \( \sim \).

So. The two categories are same (contravariant isomorphism): \( \text{aa} \)'s \( \leftrightarrow \) fg. red in \( \mathbb{A}^n \) overall \( n \) \( \sim \)

Note. In 1950's Grothendieck removed 3 hypotheses: fin gen, red, alg closed
The corresp. geom objects are affine schemes
Choose a "presentation"

\[ R = k[y_1, \ldots, y_m] / J \]

\( y_i \) generators
\( J \) relations.
\( J = \ker k[y_1, \ldots, y_m] \to R \)

\( R \) reduced \( \Rightarrow J \) radical.

Let \( Y = Z(J) \subseteq A^m \)

\( SN \Rightarrow k[Y] \cong R. \)
New varieties from old

1. Products
   - Prop. The product of aav's is an aav.

2. Complements of aav's
   - Any $V_f \subseteq \mathbb{A}^n$ aav.
   - $f \in k[V]
   - \sim V_f = V \setminus \text{Z}(f) = \{p \in V : f(p) \neq 0\}$

Examples
- $GL_n k$
- $Polyn = \{\text{polys of deg } n \text{ with distinct roots}\}$

Prop. Any $V_f$ is isomorphic to an affine variety with coord ring

$k[V_f] \cong k[V][f^{-1}] = k[V]_f$

"localization"
rat'1 functions $\frac{\text{poly}}{f^k}$
Prop. Any \( V_f \) is isomorphic to an affine variety with coord ring

\[
k[V_f] \cong k[V][f^{-1}] = k[V]_f
\]

II. Trick of Rabinowitz!

Let \( J = \mathcal{I}(V) \subseteq k[x_1, \ldots, x_n] \)

\( \mathcal{F} \in k[x_1, \ldots, x_n] \quad \mathcal{F} \in [f] \)

Set \( J_{\mathcal{F}} = (J, t\mathcal{F}-1) \subseteq k[x_1, \ldots, x_n, t] \)

We'll show \( V_f \cong W := \mathcal{Z}(J_{\mathcal{F}}) \subseteq \mathbb{A}^{n+1} \)

\[
W \leftrightarrow V_f
\]

\[
(x_1, \ldots, x_n, y) \mapsto (x_1, \ldots, x_n)
\]

Check inverses, check second statement

Example \( V = \mathbb{A}^1 \)

\[
f = x - 1
\]

\[
V_f \quad \mathbb{A}^1
\]
Chapter 2 Projective varieties. Proj space

$\mathbb{P}^n = \text{compactification of } \mathbb{A}^n$

w/ one infinitely distant pt in each direction.

→ compactification of aav's

Precisely:

$\mathbb{P}^n = (k^{n+1} - 0) / k^*$

$= (k^{n+1} - 0) / \text{nonzero scaling}$

= space of lines thru 0

so $x \sim y \iff x = \lambda y \quad \lambda \in k^*$

Write $[(x_0, \ldots, x_n)]$ as $[x_0 : \ldots : x_n]$

"homog. coords"

$n = 1$ pictures over $\mathbb{R}$

Two pics

$\mathbb{P}^1$

$\mathbb{A}^1$

identified

circle

one pt at $1$ ↔ vert line

$X_0 = 1.$

$X_1$

$X_0$

$X_1$

$X_0 = 0.$

$X_1$

$X_0$

$X_1$
algebraically:

\[ [x_0 : x_1] \]

\[ \mathbb{P}^1 = \{ [1 : x_1] \} \cup \{ [0 : 1] \} = \mathbb{A}^1 \cup \mathbb{A}^0 = \text{pt} \]

For \( k = \mathbb{C} \), \( \mathbb{P}^1_{\mathbb{C}} = \text{Riemann sphere} \).

\[ = \mathbb{C} \cup \{ \infty \} \]

For \( n = 2 \)

\[ \mathbb{P}^2 = \{ [1 : x_1 : x_2] \} \cup \{ [0 : x_1 : x_2] \}

\[ = \mathbb{A}^2 \cup \mathbb{P}^1 \]

antipodal pts id'd

\( \mathbb{R}^2 \)

\[ \mathbb{P}^1_{\mathbb{R}} \]

Also have lines in \( \mathbb{R}^2 \).

\[ \text{line thru } O \]

\[ x_0 = 1 \]

\[ n \times n \]
In general:
\[ P^n = \mathbb{A}^n \cup P^{n-1} \]
\[ = \mathbb{A}^n \cup \ldots \cup \mathbb{A}^0 \]

This decomposition is not canonical.

Let \( U_j = \{ [x_0 : \ldots : x_n] : x_j \neq 0 \} \)
\[ \implies P^n = U_j \cup U_j^{\perp} \]
\[ \mathbb{A}^n \cup P^{n-1} \]

The \( U_j \) form the standard affine cover of \( P^n \).

For \( k = \mathbb{C} \) the \( U_j \) give \( P^n \) structure of a \( \mathbb{C} \) \( n \)-manifold.

Projective subspaces

Images in \( P^n \) of linear subspaces of \( k^{n+1} \).

So a line in \( P^n \) is image of plane in \( k^{n+1} \).

Through any two pts in \( P^n \) there is a line.

Fact. Any two lines in \( P^2 \) intersect.

Pf. Any two planes in \( k^3 \) intersect.
**Projective varieties**

A projective variety in \( \mathbb{P}^n \) is a common \( \mathbb{O} \)-set of \( f_1, \ldots, f_r \in \mathbb{k}[x_0, \ldots, x_n] \) homog. if all terms have same degree.

**Fact.** The \( \mathbb{O} \)-set of \( f \)

is well def. in \( \mathbb{P}^n \).

\[ x^d f(x) = f(x x) = 0 \]

\( \iff \ f(x) = 0 \)

Note: \( \mathbb{Z}(f) \) in \( \mathbb{A}^{n+1} \) is a cone

**Examples**

1. \( \mathbb{Z}(0) = \mathbb{P}^n \)
2. \( \mathbb{Z}(1) = \emptyset \).
3. \( \mathbb{Z}(x_0, \ldots, x_n) = \emptyset \).
4. \( \mathbb{Z}(x_0, \ldots, x_n) = \{ \text{polys w/ no const term} \} \)
5. "irrelevant ideal"
6. \( \mathbb{Z}(x-a, x_0, \ldots, x_n-anx_0) = [1 : a : \ldots : an] \)
7. \( \mathbb{Z}(x_0) = "\text{hyperplane at } \infty" \)
8. \( \cong \mathbb{P}^{n-1} \)
Conics

\( X = Z(f) \)

e.g. \( f = x^2 + y^2 - z^2 \)

3 std affine charts
  - circle, hyperbola, hyperbola

Image of
\( \phi : \mathbb{P}^1 \to \mathbb{P}^3 \)

\( \phi([t_0 : t_1]) = \begin{bmatrix} t_0^3 : t_0^2 t_1 : t_0 t_1^2 : t_1^3 \end{bmatrix} \)

This is a det. variety
\( \text{rk} \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix} \leq 1 \)

\( \sim \) intersection of 3 quadrics.
“proj. rat’l normal curve of deg 3”
exercise: \( \text{Im} \phi \) is the whole variety

Tayesh: 2\textsuperscript{nd}, 3\textsuperscript{rd} cols are multiples of 1\textsuperscript{st}.
(6) $\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$

$([x_0 : x_1], [y_0 : y_1]) \mapsto [x_0 y_0 : x_0 y_1 : x_1 y_0 : x_1 y_1]$

$\text{Im } \varphi = \mathbb{Z}(z_0 z_3 - z_1 z_2)$

"quadric"

Image of "lines" on left are lines in quadric

e.g. $\varphi (\mathbb{P}^1 \times [1:0]) = \mathbb{Z} (z_1, z_3)$.

Q. Do other lines in $\mathbb{P}^1 \times \mathbb{P}^1$ map to lines? (Tong)

Future

(5) Grassmannians

$G_{r,n} = \{r\text{-dim planes in } k^n\}$

later!

(6) Products of proj. alg. vars.

later!

(7) Compact Riemann surface.

later.

(8) Moduli space.
Homogenization

\( f \in k[x_1, \ldots, x_n] \)

\( \rightsquigarrow h \in k[x_0, \ldots, x_n] \)

homog.

Just add \( x_0 \) as needed.

Example \( f(x,y) = y - x^2 \)

\( \rightsquigarrow h(x,y,z) = yz - x^2 \)

So get the old parabola

\[ [0 : 1 : 0] \]

\( \rightsquigarrow \) parabola + pt \( \rightsquigarrow \) circle.

Example 2 is also a homogenization.

Upshot: Any affine variety can be projectivized

\( \rightsquigarrow \) compactness,

more symmetry.
(Some of) HW assignment.

\[ \mathbb{C}^n \xrightarrow{\psi} \mathbb{C}^n \]

map given by elem sym polys

Surjective: FTA

Not injective: permuting roots.

Newton: these generate the invariants.

\[ (r_1, \ldots, r_n) \mapsto ( \sum_i r_i, \sum_{i < j} r_i r_j, \ldots, r_1 \cdots r_n) \]

\[ \mathbb{C}^n / \Sigma_n \]

\[ \cong \text{symm gp} \]

\[ \cong \text{of varieties.} \]

HW #1. Show \( X/G \) is aav. \( G \cup X = \text{aav.} \)

via \( k[X/G] = k[X]^G \) "invariants"

\#2. Show \( \bar{\psi} \) is an \( \cong \).
Projective closure

\( X \subseteq \mathbb{A}^n \text{ aav.} \subseteq \mathbb{P}^n \)

The proj. closure

\( \overline{X} \) is the closure of \( X \) in \( \mathbb{P}^n \) in Zariski topology

Fact? Same as Eucl. closure.

Closure: smallest closed set containing...
So proj closure: smallest proj. var containing....
( or largest homog. ideal... )

Easy: Eucl. closure \( \subseteq \) proj. closure
So: If Eucl closure is a par, it is the proj clos.

Other dir of fact?

Fact. If \( X = \mathbb{Z}_a(I) \) then
\( \overline{X} = \mathbb{Z}_p(Ih) \)
\( Ih = \) ideal gen by homog's of all elts of \( I \).

Write
\( \mathbb{Z}_a, I_a \)
\( \mathbb{Z}_p, I_p \)
to emphasize affine/proj.
Example.

\[ X_1 = \mathbb{Z}(x_2 - x_1^2) \quad X_2 = \mathbb{Z}(x_1 x_2 - 1) \]

\[ \overline{X_1} = \mathbb{Z}(x_0 x_2 - x_1^2) \quad \overline{X_2} = \mathbb{Z}(x_1 x_2 - x_0^2) \]

\( \{ \text{same!} \} \)

Extra points: Take \( x_0 = 0 \).

\[ \ln \overline{X_1} : [0 : 0 : 1] \]
\[ \ln \overline{X_2} : [0 : 0 : 1] \quad \text{and} \quad [0 : 1 : 0] \]

Not a coincidence: \( \mathbb{P}^1 \) conic in \( \mathbb{P}^2 \).

Why is \( \overline{X_1} \) actually the proj. closure.
\( \overline{X_1} \) is a proj var containing \( X_1 \)
and \( \overline{X_1} \setminus X_1 \) finite.
more \( X_i \) dense in \( \overline{X_i} \)
or \( \overline{X_i} = \) Euclidean closure of \( X_i \).

Note \( [1 : x : x^2] \to [0 : 0 : 1] \)
as \( |x| \to \infty \).

Fact. If \( X = \mathbb{Z}(f) \) then \( \overline{X} = \mathbb{Z}_p(f_h) \)

But if \( X = \mathbb{Z}_a(f, g) \) homog.
\( X \) might not be \( \mathbb{Z}_p(f_h, g_h) \)
ex. example/exercise: \( \mathbb{Z}(y - x^2, z - xy) \)
\[ \overline{X} \setminus \mathbb{Z}(wy - x^2, wz - xy) = \overline{X} u \{ w = y = 0 \} \]
Homog. Ideals

Any \( f \in k[x_1, \ldots, x_n] \)

is a sum of homog. terms

\[ f = f^{(0)} + \ldots + f^{(m)} \]

and “graded ring”

\[ k[x_1, \ldots, x_n] = \bigoplus_{d \geq 0} k[x_1, \ldots, x_n](d) \]

“homog deg \( d \) polys (union 0)"

Lemma. Let \( I \leq k[x_1, \ldots, x_n] \).

TFAE

\( \begin{align*}
\text{1) } & I \text{ gen by homog. elts} \\
\text{2) } & f \in I \Rightarrow f^{(d)} \in I \ \forall \ d.
\end{align*} \)

Such \( I \) called homog.

\[ \text{Pf} \ 1 \Rightarrow 2 \quad I = (f_1, \ldots, f_r) \quad \text{(Hilbert BT)} \]

Write \( f_i = \Sigma f_i^{(d)} \rightarrow I = (f_i^{(d)}) \).

\( 2 \Rightarrow 1 \quad I = (f_1, \ldots, f_r) \) each \( f_i \) homog.

\( (r < \infty \text{ since Noetherian}) \)

\[ f \in I \Rightarrow f = \Sigma a_i f_i \quad a_i \in k[x_1, \ldots, x_n] \]

\[ \Rightarrow f^{(d)} = \Sigma a_i^{(d-deg f_i)} f_i \in I \quad \square \]

Note. Not all elts of homog ideals are homog.

Note. A poly can always be written as \( \Sigma f_i^{(d)} \) with \( \deg f_i \) all same.

(mult. each \( f_i \) with non-max \( \deg \) by power of \( x_0 \)). \( \text{fix} \)
Fact. 1. $I_{\text{homog}} \Rightarrow \text{rad } I_{\text{homog}}$

2. Intersection, sum, product of homog. ideals is homog.

3. $I_{\text{homog}}$ then:
   
   $I_{\text{prime}} \iff \forall \text{ homog } f, g$
   
   have $(fg \in I \iff f \lor g \in I)$

(If $I_{\text{homog}}$, can test primeness only with homog elts)

Consequence: Zariski top. works.

for pav's.
Proj Nullstellensatz

Thm. k alg closed

$I \subseteq k[x_0, \ldots, x_n]$ homog.

1. $Z_p(I) = \emptyset \iff (x_0, \ldots, x_n) \subseteq \text{rad} I$

2. $Z_p(I) \neq \emptyset \Rightarrow \Pi_p(Z_p(I)) = \text{rad} I$

So:

\[
\{ \text{pav's in } \mathbb{P}^n \} \quad \iff \quad \{ \text{rad. homog. ideals in } k[x_0, \ldots, x_n] \} \quad \setminus \quad \{ \text{irrelevant ideal} \}
\]

\[\text{Pf of Thm:}\]

1. $Z_p(I) = \emptyset \iff Z_a(I) \subseteq \{0\} \iff \text{rad} I = \Pi_a Z_a(I) \supseteq (x_0, \ldots, x_n) \text{ (affine SN)}$

2. Assume $Z_p(I) \neq \emptyset$.

\[f \in \Pi_p(X) \iff f \in \Pi_a C(X) = \Pi_a Z_a(I) = \text{rad} I \text{ (affine SN)}\]

\[\square\]

Pf uses cones:

For $X \subseteq \mathbb{P}^n$ cone $C(X)$ is corresponding union of lines in $k^{n+1}$. 
Proj closure

Thm \( X \subseteq \mathbb{P}^n \subseteq \mathbb{A}^n \) an

\[ I = \mathbb{I}_a(X) \]

\[ \Rightarrow \overline{X} = Z_p(I_h) \subseteq \mathbb{P}^n \]

Pf: \[ \forall y \in \overline{X} \text{ say } G \in \mathbb{I}_p(\overline{X}) \]

\[ G \in k[x_0, \ldots, x_n] \text{ homog.} \]

\[ \Rightarrow G = 0 \text{ on } (\overline{X} \cap U_0) = \overline{X} \setminus \{x_0 \neq 0\}. \]

\[ \Rightarrow g = G|_{x_0=1} \text{ is } 0 \text{ on } X \]

\[ g \in k[x_1, \ldots, x_n] \]

\[ \Rightarrow g \in \mathbb{I}_a(X) = I \]

\[ \Rightarrow g_h \in I_h \]

\[ \Rightarrow G = g_h x_0^t \text{ some } t. \]

\[ G = x_0^3 x_1 + x_0^2 x_1 x_2 + x_0^4 \]

\[ g = x_1 + x_1 x_2 + 1 \]

\[ g_h = x_0 x_1 + x_1 x_2 + x_0^2 \]

\[ \Rightarrow G \in I_h \text{ (since } g_h \in I_h \text{).} \]

Thus \( \mathbb{I}_p(\overline{X}) \subseteq I_h \) \( \because \overline{X} \text{ closed.} \)

\[ \Rightarrow Z_p(I_h) \subseteq Z_p \mathbb{I}_p(\overline{X}) = \overline{X} \]

\( \square \)
**Example**

\[
x = \mathbb{Z}(x, y-x^2) = \{0\} \leftrightarrow [1:0:0] \text{ in } \mathbb{P}^2
\]

\[\bar{x} = x \mapsto Uz\]

\[\not\mathbb{Z}(x, y^2-x^2) = \{[1:0:0], [0:0:1]\}\]

\[\text{Cor. } X = \mathbb{Z}(f) \Rightarrow \bar{X} = \mathbb{Z}(f_h)_p\]

**Pf.** (f) = \{fg : g \in k[x_1, \ldots, x_n]\} \quad f_h

\[\Rightarrow \bar{X} = \mathbb{Z}_p((fg)_h : g \in k, (x, \ldots, x_n))\]

\[= \mathbb{Z}_p(f_h g_h : g \in k, [x, \ldots, x_n])\]

\[= \mathbb{Z}_p(f_h) \quad \square\]

**Cor of Proj Null**

\[
\begin{align*}
\text{irred. proj vars} & \quad \leftrightarrow \quad \text{irred affine vars} \\
Y \subseteq \mathbb{P}^n & \quad \leftrightarrow \quad X \subseteq \mathbb{A}^n \\
Y \not\subseteq \mathbb{Z}(x_0) & \quad \Rightarrow \quad \bar{X} \mapsto X \subseteq \mathbb{A}^n \subseteq \mathbb{P}^n \\
Y & \mapsto Y \cap U_0 \subseteq \mathbb{A}^n
\end{align*}
\]

Why you need irreducible: (Toussel)

\[
X_0 X_2 - X_1^2 \\
X_0 X_2 - X_1^2 X_6
\]

**Pf hint:** polys \leftrightarrow polys. (homog & dehomog).
Morphisms

Naive defn: polyn. maps.

Example $C = \mathbb{Z}(xz - y^2)$

$\phi: \mathbb{P}^1 \rightarrow C \subseteq \mathbb{P}^2$

$[s:t] \mapsto [s^3:st:t^2]$

- $\phi$ is well def
- $\text{im } \phi = C$

(This is a Veronese map)

In $U_t$ chart, set $u = s/t$

$u \mapsto (u^2, u) \in U_z$

In $U_s$: $v \mapsto (v, v^2) \in U_x$

These are affine morphisms.

Now for other direction...

$\psi: C \rightarrow \mathbb{P}^1$

$[x:y:z] \mapsto \begin{cases} [x:y] \text{ on } U_x \\ [y:z] \text{ on } U_z \end{cases}$

Defined on all of $C$: $x = z = 0 \Rightarrow y = 0$.

Well def. on $C$: $x, z \neq 0 \Rightarrow y \neq 0$ so

$[x:y] : [yx:y^2] = [xy:xz] = [y:z]$

On $U_x, U_z$: $\psi$ is affine morphism

but $\psi$ is not globally polynomial.

No way to write $\psi$ as $[f_1:f_2]$

(exercise?)
Aside: Stereographic proj.

The map \( \psi \) can be defined as follows.

Let \( Q = [1:0:0] \in C \) (pt at \( \infty \))

\[ L = Z(x^2 + y^2) \]

line in \( \mathbb{P}^2 \)

For \( P = [a:b:c] \in C \), \( P \neq Q \)

The line \( PQ \) is \( yc = zb \)

and \( PQ \cap L = \psi(P) = [0:b:c] \)

We want (need?) this to a morphism, but not a poly.
From last time

Example: \( C = \mathbb{Z}(xz - y^2) \)

\[ q: \mathbb{P}^1 \to C \subseteq \mathbb{P}^2 \\
[s: t] \mapsto [s^2: st: t^2] \]

\[ \psi: C \to \mathbb{P}^1 \]

\[ [x: y: z] \mapsto \begin{cases} [x: y] \text{ on } U_x \\ [y: z] \text{ on } U_z \end{cases} \]

Today: Morphisms, birational maps

Correspondence

PAV's \( \leftrightarrow \) extensions of \( k \).

Morphisms of PAVs

\( V \subseteq \mathbb{P}^n, W \subseteq \mathbb{P}^n \) pav's

\( f: V \to W \) is a morphism if

\( \forall p \in V \exists \text{ homog polys } f_0, \ldots, f_m \in k[x_0, \ldots, x_n] \) s.t. for some nonempty open nbhd of \( p \)

\( f|_U \) agrees with

\[ U \to \mathbb{P}^m \\
q \mapsto [f_0(q): \ldots : f_m(q)] \]

e.g. \( q, \psi \) above.
Morphisms of \( \mathbb{PAVs} \)

\( V \subseteq \mathbb{P}^n, \ W \subseteq \mathbb{P}^n \) pav's

\[ f: V \rightarrow W \text{ is a morphism if} \]

\( \forall \ p \in V \ \exists \ \text{homog polys} \]

\( f_0, \ldots, f_m \in k[x_0, \ldots, x_n] \) s.t. for

some nonempty open nbhd of \( p \)

\[ f|_U \text{ agrees with} \]

\( U \rightarrow \mathbb{P}^n \)

\[ q \rightarrow [f_0(q) : \ldots : f_m(q)] \]

e.g. \( \varphi, \psi \) above.

Notes

1. Can also allow rat'l fn's (clear denoms).

2. To have a well-def map, \( f_i \) must have same deg.

3. Also, \( f_i \) must not all vanish at a single pt.

4. Implicit: different fn's agree on overlaps (since \( f \) globally def).

Isomorphism: if \( f \) inverse morphism.
Examples

1. $\phi$, $\psi$ above

   $\mathbb{P}^1 \rightarrow C = \text{parabola}$

   are isomorphisms

   e.g. $[s:t] \mapsto [s^2:st:t^3]$

   $\psi|_{Ux} \mapsto [s^2:st] = [s:t]$

2. Any homog rat'f $h: X \rightarrow k$
   can be considered a morphism $\phi, \psi$ homog

   $\phi: X \rightarrow \mathbb{P}^1$. If $h = f/g$, same deg

   $\phi([x_0: \ldots : x_n]) = [f(x_0, \ldots, x_n) : g(x_0, \ldots, x_n)]$

3. Linear change of coords on $\mathbb{P}^n$.
   Later: All isoms $\mathbb{P}^n \rightarrow \mathbb{P}^n$ are of this form.

   Conseq 1. $H = \text{hyperplane in } \mathbb{P}^n$
   $\Rightarrow H \cong \mathbb{P}^{n-1}$

   (using: restriction of $\cong$ is $\cong$)

   Conseq 2. All conics in $\mathbb{P}^2$ are $\cong$.

   Conics $\leftrightarrow$ symm bilin $\leftrightarrow$ quad $\leftrightarrow$ $3 \times 3$ matrices

   $Z(x^2 + 4xy + 3y^2) \leftrightarrow (x, y, z) \leftrightarrow \begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} (x, y, z)$

   But: All symm. $\mathbb{C}$-matrices diagable...
Coord ring of PAVs

Can define:

\[ k[X] = k[x_0, \ldots, x_n] / \mathbb{P}_p(X) \]

Issue #1. The elts of \( k[X] \)
don't give well-def fn's on \( X \).
In fact: Every rat fn on \( X \)
is const.

For \( k = \mathbb{C} \) this is Liouville's thm
(bounded holom fn's are const)
plus fact that \( X \) compact.

Issue #2. Can have \( X \cong Y \)

\[ k[X] \neq k[Y] \]

E.g. \( X = \mathbb{P}_p(x) \subseteq \mathbb{P}^2 \sim k[X] \cong k[y, z] \)
\( Y = \mathbb{P}_p(x^2+y^2-z^2) \cong \mathbb{P}^1 \) UFD

\[ \sim k[Y] \cong k[x, y, z] / (x^2+y^2-z^2) \]
not UFD.

\( z \cdot z = (x+iy)(x-iy) \)

Fixes:
1. \( k(Y) \)
2. rational maps
Rational maps

\( X \subseteq \mathbb{P}^n, \ Y \subseteq \mathbb{P}^m \)

A rational map

\( \varphi : X \dashrightarrow Y \)

is an eq class of expressions

\([f_0 : \ldots : f_m] \) s.t.

1. \( f_0, \ldots, f_m \in k[x_0, \ldots, x_n] \)
   homog. of same deg.

2. \([f_0(p) : \ldots : f_m(p)] \neq [0 : \ldots : 0]\)
   some \( p \in X \)

3. \( \forall p \in X \) if \([f_0(p) : \ldots : f_m(p)]\)
   is defined, it is in \( Y \).

Two expressions are equiv. if they are equal where both defined.

example: \( \varphi, \psi \) from start of class:

\([x:y] \mapsto \{ [x:y:z] \text{ on } U_x \}
\{ [y:z] \text{ on } U_z \}

Q. Why transitive?

Say \( \varphi \) is regular at \( p \) if

\( \varphi(p) \) defined for some expression representing \( \varphi \).

So \( \varphi \) is not defined at

non-regular pts.

Rat. maps are like morphisms, but
only def on open subset of \( X \).
**Example** Cremona transformation.

\[ \varphi: \mathbb{P}^2 \rightarrow \mathbb{P}^2 \]

\[ [x:y:z] \mapsto [yz:xz:xy] \]

Not def. at \([0:0:1]\) or any pt with two zeros.

In other words: \( \mathbb{Z}(x,y) \cup \mathbb{Z}(y,z) \cup \mathbb{Z}(x,z) \)

---

Problem with rat'l maps:

can't rec. compose \(f \circ g\)

if \(g(\text{dom } g) \cap \text{dom } f = \emptyset\).

**Dominant maps**

\[ \varphi: X \rightarrow Y \text{ is dominant} \]

if \(\varphi(\text{dom } \varphi)\) nonempty, open in \(Y\)

If \(\varphi\) dominant, can compose \(\psi \circ \varphi\)

**Example** Cremona map is dominant

and \(\varphi \circ \varphi \simeq \text{id.}\)

\[ [x:y:z] \mapsto [yz:xz:xy] \mapsto [x^2yz:xy^2z:xyzt^2] = [x:y:z] \text{ if } x,y,z \neq 0. \]
**Field of rat'l fns**

\[ k(X) = \{ \frac{f}{g} : f, g \in k[x_0, \ldots, x_n] \text{ homog of same deg} \} / \sim \]

\[ f_1/g_1 \sim f_2/g_2 \text{ if } f_1g_2 - g_1f_2 \in \mathbb{I}_P(X) \]

~ Well-def fns on \( X \) (open subset of \( X \))

\[ g \neq 0. \]

**Thm**

A rat'l map \( \phi : X \rightarrow Y \) is birational (has rat'l inverse) if \( \phi \) dominant & \( \phi^* \text{ is } \sim \)

\[ \iff \phi \text{ dominant} \& \phi^* \text{ is } \sim \]

2. \( X, Y \) birat. equiv \iff \( k(X) \cong k(Y) \)

So: equiv of categories

\[ \{ \text{irred. quasi-proj. vars} \} \leftrightarrow \{ \text{field exts of } k \text{ w/ birat. maps w/ } k\text{-homoms.} \} \]
From last time...
A pos. criterion for dominance

Lemma. \( \varphi : X \to Y \) ratio map btw proj vars, \( Y \) irreducible. If \( \exists Z \subseteq Y \) par s.t. \( \text{im } \varphi \) contains \( Y \setminus Z \) then \( \varphi \) is dom.

Defn of dominant: \( \text{im } \varphi \) not contained in subvar (assuming \( Y \) irreducible).

Proof of Lemma. Follow your nose.
Contradict irreducibility.

\[ \text{image} \]

\[ \text{open} \]

??

\[ \text{dense.} \]

\[ \text{in } Z \text{ subsp. top.} \]

(From last time)
Chap 3 Classical constructions
(Veronese, Segre, Grassmannian).

Veronese Maps

$k[x_0, \ldots, x_n](d) = \{ \text{deg } d \text{ homog } \}
\subseteq k\text{-vect sp on the } \binom{d+n}{n}
\text{monomials of deg } d.

V_d: \mathbb{P}^n \rightarrow \mathbb{P}^m 
[\ldots [x_0 : \ldots : x_n] \mapsto [x_0^d : \ldots ]
\text{all deg } d \text{ monomials}
in x_0, \ldots, x_n.

- $V_d$ is well def \( \checkmark \)
  (all deg $d$, don't all vanish)

- $V_d$ is injective.
  look at $x_0^{d-1} X_i$ coords.
  where $X_0 \neq 0$.

\[ \begin{bmatrix} x_0^d : x_0^{d-1} : x_1 : x_0^{d-1} : x_2 : \ldots \end{bmatrix}
\sim \begin{bmatrix} x_0 : x_1 : x_2 \ldots \end{bmatrix} \]
Examples

1. \( n=1, d=2 \)
   \[ V_2 : \mathbb{P}^1 \to \mathbb{P}^2 \]
   \[ [s:t] \mapsto [s^2: st: t^3] \]
   \( W_{1,2} = \text{im} \, V_2 = \mathbb{Z}(xz-y^2) \)
   & \( V_2 \) is \( \cong \) onto image.

2. \( n=1, d=3 \)
   \[ V_3 : \mathbb{P}^1 \to \mathbb{P}(4) - 1 = \mathbb{P}^3 \]
   \[ [s:t] \mapsto [s^3: s^2t: st^2: t^3] \]
   \( W_{1,3} = \text{im} \, V_3 = \text{rat.'l normal curve of deg} \, 3 \)
   = proj clos. of twisted cubic:

\[ W = \mathbb{Z}(xw-y^2, y^2-xz, wy-z^2) \]

Easy: \( \text{im} \, V_3 \subseteq W \)

Hard: \( W \subseteq \text{im} \, V_3 \) (Arrondo)
Chris: maybe direct? Proj vers.

3. \( n=1, d \)
   \( \text{im} \, V_d = \text{rat.'l norm. curve of deg} \, d \)
   = Vanish. set of \( 2 \times 2 \) det's
   \( (Z_0, d \quad Z_1, d-1 \ldots Z_{d-1}, 1) \)
   \( (Z_1, d-1 \quad Z_2, d-2 \ldots Z_{d}, 0) \)

\[ Z_{ij} \leftrightarrow s^j t^i \quad i+j = d \]

Check: \( Z_{i,j} \cdot Z_{k,l} = Z_{i+k, j+l} \)
4. Veronese surface

\[ \nu_2 : \mathbb{P}^2 \rightarrow \mathbb{P}(\frac{4}{2}) \cong \mathbb{P}^5 \]

\[ [s : t : u] \mapsto [s^2 : t^2 : u^2 : st : su : tu] \]

\( \text{Im} \nu_2 \) is van set for 2x2 minors of

\[ \begin{pmatrix} z_0 & z_3 & z_4 \\ z_3 & z_1 & z_5 \\ z_4 & z_5 & z_2 \end{pmatrix} \] (rank 1 condition)

For general deg 2:

\( \text{Im} \nu_2 \) = van set for 2x2 minors of

\[ \begin{pmatrix} Z_{i,j} \end{pmatrix} \text{ symmetric} \]

\[ Z_{i,j} \leftrightarrow X_{i-1} X_{j-1} \]
Image of Veronese

Let $W = \text{im } Vd (= V_d, d)$, $I = (i_0, \ldots, i_n)$

Let $x^I \leftrightarrow x_0^{i_0} \cdots x_n^{i_n} \leq i_j = d$

Prop. $W$ is vanish. set of

$$\{ x^I x^J - x^K x^L : I + J = K + L \}$$

Q. Can this be written in terms of determinants?

A. Yes? $n+1$ rows $d+1$ cols

Q. Proof of Prop?

Prop. $V_d : \mathbb{P}^n \rightarrow W$ is $\cong$ onto image.

Pf. Construct inverse.

On each pt of $W$ at least one $x_i^d$ is nonzero.

$\rightarrow Ux_i$ cover $W$

Define $Ux_i \rightarrow \mathbb{P}^n$

$x \mapsto "x_j x_i"$ coords

as in proof of injectivity.

These agree on overlaps, give inverse to $Vd$
A possible hint for proving the

Prop:
\[ \Theta : k[x^I] \longrightarrow k[x_0, \ldots, x_n] \]

Show \( \ker \Theta \) is generated by the \( x^I x^J - x^k x^L \).
Hypersurface Sections

\( f = \text{nonzero poly of deg } d \geq 1. \)
\( \leadsto Z(f) = \text{hypersurf of deg } d. \)

For \( X = \text{par}, Z(f) \cap X \) called a hypersurf. section.

Thm. \( X \setminus (Z(f) \cap X) \) is an affine alg var. (if not \( \emptyset \))

Application.

\( \text{Polyn}_{/\sim} = \{ \text{polys of deg } n \text{ with } \exists / \text{scale}. \}
\)
\( \text{nonzero discriminant} \)

is affine. \( \downarrow \text{homog: } \Pi(X_i - X_j) \)

\( \text{Pf for } d = 1 \quad Z(f) = \text{hyperplane}, \)
\( \text{WLOG } x_0 = 0. \)

\( \text{Pf of general } d \quad \text{Apply } \text{Vd} \)

hypersurf \( Z(f) \leadsto \text{hyperplane}. \)
apply \( d = 1 \) case. use fact that \( \text{Vd (variety)} = \text{proj var} \)

(next page).

example \( . \quad f = x^2 - 3yz \leq \mathbb{P}^2 \)

This \( (x^2) - 3(yz) \) in Veronese coords \( \leadsto \text{linear!} \)
Images of varieties

Prop. \( X \subseteq \mathbb{P}^n \) is pav

\[ \Rightarrow V_d(X) \] is pav any \( d \).

\[ \text{If by example (Hamm's)} \]

\[ X = \mathbb{Z}(x_0^3 + x_1^3 + x_2^3) \]

multiply by all \( x_i \) to

get 3 polys of deg 2 \times 2

\[ X = \mathbb{Z}(x_0^4 + x_0 x_1^3 + x_0 x_2^3, \]

\[ x_1 x_0^3 + x_1^4 + x_1 x_2^3, \]

\[ x_2 x_0^3 + x_2 x_1^3 + x_2^4) \]

Apply \( V_2 \rightarrow 3 \) quadratics.

But \( \text{Im} V_2 \) is van set of 6 quadratics.

So \( \text{im } X \) is van set of 9 quadratics.
Q. Suppose $Z \subseteq \mathbb{P}^n$ dense & $Z =$ image of a morphism on q.p. var

Then $Z$ open?

Optional homework:
1. above
2. Image of $Vd =$
   $Z(x^I x^J - x^K x^L)$
Segre Map

Goal: products of par's are par's.

easy for affine space since $\mathbb{A}^m \times \mathbb{A}^n = \mathbb{A}^{m+n}$

Note $\mathbb{P}^m \times \mathbb{P}^n$ not even homeo to $\mathbb{P}^{m+n}$

Identify $\mathbb{P}^{(m+1)(n+1)-1}$ with $M_{m+1,n+1}(k)/\text{scalar}$.

Define $\varphi_{m,n}: \mathbb{P}^m \times \mathbb{P}^n \rightarrow \mathbb{P}^{(m+1)(n+1)-1}$

\[
([x_0: \ldots : x_m], [y_0: \ldots : y_n]) \mapsto \\
\left(\begin{array}{c}
\frac{x_0y_0}{x_0y_0} \\
\vdots \\
\frac{x_my_0}{x_my_0} \\
y_1 \ldots y_n
\end{array}\right) = \\
\left(\begin{array}{c}
x_0 \\
\vdots \\
x_m \\
y_1 \ldots y_n
\end{array}\right)
\]

$\text{Im } \varphi_{m,n} = \text{Segre variety, "outer product"}$

Use homog coords $Z_{ij} \leftrightarrow x_i y_j$
Example \( \varphi_{1,1} : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3 \)

\[
([x_0 : x_1], [y_0 : y_1]) \mapsto \left[ \begin{array}{c} x_0 y_0 \\ x_0 y_1 \\ x_1 y_0 \\ x_1 y_1 \end{array} \right]
\]

Note: \( \det = 0 \implies \operatorname{rk} \leq 1 \).
Also \( \operatorname{rk} \neq 0 \implies \operatorname{rk} = 1 \)
Thus \( \varphi_{1,1} \) well def &

\[ \operatorname{Im} \varphi_{1,1} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \det = 0 \right\} \]

(in lin alg: all rank 1 matrices are outer products)

Claim: \( \varphi_{1,1} (\mathbb{P}^1 \times pt) \) is linear

\( \iff \) plane in \( \mathbb{P}^4 \)

\( \mathbb{P}^1 \times [1 : b] \)

\[
\varphi_{1,1} \left( \begin{array}{c} x_0 \\ x_1 \end{array} \right) = \begin{pmatrix} x_0 b & x_0 \\ x_1 b & x_1 \end{pmatrix}
\]

\[ Z_{01} = b Z_{00}, \quad Z_{11} = b Z_{10} \]

Intersection of 2 3-planes in \( \mathbb{P}^4 \).
Prop. \( \varphi_{m,n} \) injective.

**Pf.** Let \( M = (m_{ij}) = \varphi_{m,n}(a,b) \)

WLOG \( a_0 = b_0 = 1 \Rightarrow m_{00} = 1 \)

Recover \( a, b \) from first col, row resp. \( \square \)

Prop. \( \text{Im} \ \varphi_{m,n} = \{ \text{rank 1 matrices} \} / \text{scale}. \)

**Pf.** Use above lin alg fact or:

Say \( \text{rk} \) of \( M = (m_{ij}) = 1 \)

Scale so \( m_{00} = 1 \)

\( \forall \ k, l \neq 0 \ \ m_{kl} = m_{00}m_{kl} \) (is \( m_{00} \) 1st)

Take \( a, b \) to be first col/row.

---

**Algebraic structure on** \( \mathbb{P}^m \times \mathbb{P}^n \)

\( \varphi_{m,n} \) gives \( \mathbb{P}^m \times \mathbb{P}^n \) an alg.

structure:

- varieties in \( \mathbb{P}^m \times \mathbb{P}^n \)
  are intersections of vars
    in \( \mathbb{P}^N \) with \( \text{Im} \ \varphi_{m,n} \)
    (subspace topology)

- poly fsns on \( \mathbb{P}^m \times \mathbb{P}^n \)
  are poly fsns on \( \text{Im} \ \varphi_{m,n} \)
Prop. Under this defn, subvarieties of $\mathbb{P}^n \times \mathbb{P}^n$ are zero sets of bihomog. polys.

* if $x_i, y_i$ are coords on $\mathbb{P}^n, \mathbb{P}^n$, each monomial has fixed deg in $x_i$ & fixed deg in $y_i$. If the deg's are same, say the bihomog. poly is balanced.

Pf. Given subvar of Segre var: $Z(f_1, \ldots, f_r)$
Each $f_i$ pulls back to balanced poly in $x, y$. If $\deg f_i = d_i$, pullback has bi-degree $(d_i, d_i)$
eq \varphi_{m,n}(Z_{00} - Z_{01}Z_{02}) = (x_0y_0)^2 - (x_0y_1)(x_0y_2)$

\[ Z(\varphi_{m,n}(f)) \rightarrow \varphi_{m,n} \downarrow \quad \text{pullback} \]

\[ \text{subvars} \quad \text{subvars} \]
Other direction: Given 
\( f_1, \ldots, f_r \) bihomog. in \( x_i, y_i \)
can make each balanced
w/o changing zero set (cf last lecture):
replace \( f_i \) with
\[ \{ y_0 f_i, \ldots, y_n f_i \} \]

Notice: There are many more 
varieties in \( \mathbb{P}^m \times \mathbb{P}^n \) than
just products of varieties:
product of vars \( \leftrightarrow \) polys factorable as
\( (\text{poly in } x) \cdot (\text{poly in } y) \).

• Another way to define products 
of proj vars:
\[ k[x \times y] = k[x] \otimes k[y] ? \]
(Probably
Maybe with \( k(x) \)?)

\[ X \times Y \text{ is a categorical product} \]
(satisfies univ property).
Given \( \ell_x : Z \rightarrow X \)
\( \ell_Y : Z \rightarrow Y \)
\[ f \quad \ell : Z \rightarrow X \times Y \]
s.t. \( \Pi_x \circ \ell = \ell_x \) same for \( y \).
Example. Twisted cubic.

\[ C = \text{image of} \quad [s:t] \mapsto [s^3:s^2t:st^2:t^3] \]

Observe \( C \subseteq \text{Segre}_{1,1} \subseteq \mathbb{P}^3 \).

\[ \det (s^3, s^2t, st^2, t^3) = 0. \]

Besides the eqn defining Segre_{1,1}, there are 2 polys defining \( C \) in \( \mathbb{P}^3 \):

1. \( \mathbb{Z}_{00} \mathbb{Z}_{10} - \mathbb{Z}_{0}^2 \)
2. \( \mathbb{Z}_{01} \mathbb{Z}_{11} - \mathbb{Z}_{10}^2 \)

1. Pulls back to \( C \) union a line:

\[ x_0 y_0 x_1 y_0 - (x_0 y_1)^2 \]

\[ = x_0 (y_0^2 x_1 - x_0 y_1^2) \underset{\text{line } x_0 = 0}{\longleftrightarrow} \mathbb{A} \mathbb{Z}(f) \]

Check: \( q_{1,1} \) maps \( \mathbb{A} \mathbb{Z}(f) \) bij to \( C \).
Coord-free descriptions of \( V_d \) & \( \varphi_{m,n} \)

Similarly \( \varphi_{m,n} \) comes from 
\[
K^{m+n} \times K^{n+1} \rightarrow (K^m)^{\otimes (k^{n+1})}
\]

3 natural map
\[
k^{n+1} \rightarrow \text{Sym}^d(k^{n+1})
\]

\( v \mapsto v^d \)

Projectivizing gives \( V_d \)

e.g. \( \nu_1 : \mathbb{P}^1 \rightarrow \mathbb{P}^2 \)
\[
k^2 \rightarrow \text{Sym}^2 k^2
\]
\[
e_1, e_2 \quad \rightarrow \quad e_1^2, e_1 e_2, e_2^2
\]
\[
(xe_1 + ye_2) \mapsto (xe_1 + ye_2)^2 = x^2(e_1^2) + xy(e_1 e_2) + y^2(e_2^2)
\]

\[
\text{Sym}^d(V) = V^{\otimes d}/\text{rearranging terms}
\]
Grassmannian

\[ V = k^n \]
\[ \text{Gr}_r_n = \text{Gr}_r(V) = \{ \text{r-dim subsps of } V \} \]
e.g. \[ \text{Gr}_{1,n} = \mathbb{P}^{n-1} \]

Today: \( \text{Gr}_r_n \) is a proj av.

So: The "moduli/parameter space of r-dim lin. varieties is a variety"

Topology aside
\[ B = \text{space.} \]
An r-plane bundle is a (bigger) space so "over" each \( b \in B \), have r-plane.

Examples:
1. \( B = S^1 \) \( r = 1 \)

\[ S^1 \times \mathbb{R} \]
open annulus.

2. \( M = \text{Smooth manifold} \)
\[ TM = \text{r-plane bundle over } M \]
open Mobius band.
Amazing fact:

\[ \{ r\text{-bundles over } B \} \overset{\sim}{\leftrightarrow} \{ B \to \text{Gr}, \infty \} \]

Why? \( \text{Gr}, n \) (and \( \text{Gr}, \infty \)) have a canonical \( r \)-plane bundle \( E \) over them.

\[ E \subseteq \text{Gr}, n \times k^n \]

\[ \{ (W,v) : v \in W \} \]

Example. \( G_{1,2} \; k = \mathbb{R} \).

and given \( B \to \text{Gr}, n \)

can pull back the bundle over \( \text{Gr}, n \).
Back to the goal: $\text{Gr}, n$ is par.

**Direct approach**

We define $\text{Gr}, n \rightarrow \mathbb{P}^{(\frac{n}{r})-1}$

Given $W \in \text{Gr}, n$

$\twoheadrightarrow$ basis $V_1, \ldots, V_r$

$\twoheadrightarrow$ $r \times n$ matrix

$\twoheadrightarrow$ $\binom{n}{r}$ minors $\in \mathbb{K}^{(\frac{n}{r})}$

Different bases give $r \times n$ matrices that differ by mult on left by invertible $r \times r$ matrix $A$

This changes all minors by $\det A$.

$\leadsto$ well def pt in $\mathbb{P}^{(\frac{n}{r})-1}$.

Need to show:

- injective
- image is variety.

For latter, show the image satisfies Plücker relations:

Denote by $M_{i_1 \ldots i_r}$ the minor...

Given $i_1 < \ldots < i_{r-1}$

$j_1 < \ldots < j_{r+1}$

$$0 = \sum_{l=1}^{r+1} (-1)^l M_{i_1 \ldots i_{l-1} j_l} M_{i_1 \ldots i_{l-1} j_{r+1}}$$

$\leadsto$ many quadrics
Examples

\[ W \in G_{1,3} \quad W = \text{Span} \{(\frac{a_0}{a_1}, \frac{a_2}{a_2})\} \]

\[ \sim (a_0, a_1, a_2) \]

minors: \( a_0, a_1, a_2 \).

\[ W \in G_{2,3} \quad W = \text{Span} \{a, b\} \]

\[ \sim (a_0, a_1, a_2) \]

\[ \sim (b_0, b_1, b_2) \]

minors \( \leftrightarrow \) cross product.

\[ \text{Pl"ucker: } (-1) M_{01} M_{02} \]

\[ \text{Can see injectivity in both cases.} \]

Observation (1on): \( G_{1,n} \cong G_{n-1,n} \)

\[ G_{r,n} \cong G_{n-r,n} \]

First nontrivial Pl"ucker relation: \( G_{2,4} \)

\[ M_{12} M_{34} - M_{13} M_{24} + M_{14} M_{23} \]

single defining poly.

Can see injectivity in both cases, \& surjectivity to \( \mathbb{P}^2 \).
Second approach: Wedge products

\[ V = \text{vect sp. over } k \]

\[ V \otimes r = V \times \ldots \times V \text{ /multilinearity.} \]

\[ = \{ \text{finite sums of } v_1 \otimes \ldots \otimes v_r \} \]

subject to

\[(av_1 + a'v_1') \otimes v_2 \otimes v_3 \]

\[ = a v_1 \otimes v_2 \otimes v_3 + a' v_1 \otimes v_2 \otimes v_3 \]

Why? \{multilinear maps \( V^r \rightarrow W \}\]

\[ \leftrightarrow \{ \text{linear maps } V^\otimes r \rightarrow W \} \]

Next...

\[ \Lambda^r V = V \otimes r \text{ /alternating.} \]

\[ = \{ \text{finite sums } v_1 \Lambda \ldots \Lambda v_r \} \]

subject to multilinearity as above and: swapping two entries gives -1

So: \[ v_1 \Lambda v_2 \Lambda v_3 = -v_2 \Lambda v_1 \Lambda v_3 \]

and \[ v_1 \Lambda v_1 \Lambda v_2 = -v_1 \Lambda v_1 \Lambda v_2 \]

\[ \Rightarrow v_1 \Lambda v_1 \Lambda v_2 = 0 \]

(char \( k \neq 2 \))
Why?
1. \{alt. multilin. maps \( V^r \rightarrow W \}\) 
\[\Leftrightarrow \{\text{lin maps } \Lambda^r V \rightarrow W \}\]
2. \( \Lambda^r k^n \cong k \) \( \Rightarrow \) determinants exist and are unique.
3. Area functions in \( k^n \)
   \[
   (e_1 + e_2) \wedge e_3 = e_1 \wedge e_3 + e_2 \wedge e_3
   
   \text{area of proj to } (e_1 + e_2)e_3 \text{ plane } = \text{proj to } e_1e_3 \text{ plane } + \text{proj to } e_2e_3 \text{ plane}
   
   \text{where } e_1 + e_2 \text{ declared to have length } 1
   
Facts
1. If \( v_1, \ldots, v_n \) basis for \( V \) then \( \{v_{i_1} \wedge \cdots \wedge v_{i_r} : 1 \leq i_1 \leq \cdots \leq i_r \} \)
   is a basis for \( \Lambda^r V \)
   \[\Rightarrow \dim \Lambda^r V = \binom{n}{r}\]
2. \( W \subseteq V \) subsp of dim \( r \)
   \[T \in \text{Aut}(W) \]
   \[w \in \Lambda^r W \]
   \[\Rightarrow T(w) = (\det T) w\]
Plücker embedding \( F : \text{Gr}_{r,n} \rightarrow \mathbb{P}(\Lambda^r V) = \mathbb{P} \)

\( \Span \{v_1, \ldots, v_r\} \mapsto [v_1 \wedge \cdots \wedge v_r] \)

Well def by Fact 2

e.g. \( v_1 \wedge v_2 = (v_1 + v_2) \wedge v_2 \)

\( = v_1 \wedge v_2 + v_2 \wedge v_2 \)

5v_1 \wedge v_2 \sim v_1 \wedge v_2

Todo: \( F \) inj

. \( \text{Im} F \) is proj var.

Will do at same time.

\[ \text{Defn. } x \in \Lambda^r V \text{ is totally decomposable.} \]

if it's an \( r \)-wedge (not a sum)

\[ \text{Note: } \text{Im} F = \{ \text{totally dec} \} \]

\( e_1 \wedge e_2 + e_3 \wedge e_4 \) is the simplest example of not-\((\text{tot. dec})\)

\[ \text{Lemma. Given nonzero } x \in \Lambda^r V \]

Let \( \varphi_x : V \rightarrow V^{r+1} V \)

\( v \mapsto v \wedge x \)

1. \( \dim \ker \varphi_x \leq r \), with \( = \) iff \( x \) tot. dec.

2. If \( x = v_1 \wedge \cdots \wedge v_r \) then \( \ker \varphi_x = \Span\{v_{r+1}, v_r\} \).
Lemma. Given nonzero $x \in \mathbb{N}^r V$

Let $\varphi_x : V \to \mathbb{N}^{r+1} V$

$$v \mapsto v \wedge x$$

1. $\dim \ker \varphi_x \leq r$, with $x \totdec$ iff $x \totdec$.
2. If $x = v_1 \wedge \ldots \wedge v_r$ then $\ker \varphi_x = \Span\{v_1, \ldots, v_r\}$

---

1. $\Rightarrow F \text{ inj.}$
2. $\Rightarrow \text{im} F$ is a variety because:

- $x \in \text{im} F \iff x \totdec$.
- Nullity $\varphi_x \geq r$.
- $\Rightarrow \text{rank } \varphi_x \leq n-r$
- $\Rightarrow$ all $n-r+1$ minors vanish.

$G_{r,n} \to \mathbb{N}^r V$

$\Span\{v_1, \ldots, v_r\} \to v_1 \wedge \ldots \wedge v_r$

$\mathcal{P}(\mathbb{N}^r V) \to \mathcal{P}(\text{Hom}_k(V, \mathbb{N}^{r+1} V))$

$x \mapsto \varphi_x$

Inj. & linear, can apply $\mathcal{P}$

rank $\leq n-r$ defines closed subset of RHS

$\leadsto$ closed subset of RHS

$\leadsto$ closed subset of $\mathcal{P}(\mathbb{N}^r V)$ (preim. of closed is closed).
Grassmannians

$Gr_{r,n} = \{ r \text{-planes in } V = K^n \}$

Goal: this is proj. alg var.

Plücker embedding

$F : Gr_{r,n} \rightarrow P(\Lambda^r V)$

$\text{Span \{v_1, \ldots, v_r\}} \rightarrow [v_1 \wedge \ldots \wedge v_r]$

To show: ① $F$ inj
            ② $\text{Im } F$ closed.

Note: $\text{Im } F = \{ \text{tot. dec. elts} \}$

Tool: Wedging map

$x \in \Lambda^r V$

$\rightarrow \varphi_x : V \rightarrow \Lambda^{r+1} V$

$v \rightarrow v \wedge x$

Have $\varphi_x \in \text{Hom}_k(V, \Lambda^{r+1} V)$

Lemma. $x \in \Lambda^r V$, $x \neq 0$.

Then ① $\dim \ker \varphi_x \leq r$.

$\text{Im } F$ closed $\iff$ ② equality $\iff x$ tot. dec.

$F$ inj $\iff$ ③ if $x = a \cdot v_1 \wedge \ldots \wedge v_r$ tot dec

$\ker \varphi_x = \text{Span \{v_1, \ldots, v_r\}}$
Lemma. \( x \in \Lambda^r V, x \neq 0. \)

Then
1. \( \dim \ker \varphi_x \leq r. \) \text{ Given} 
2. \( \text{equality} \iff x \text{ tot. dec.} \iff \rk \varphi_x \leq n-r \)
3. \( \text{If } x = a \cdot v_1 \wedge \ldots \wedge v_r \text{ tot dec} \)
   \( \ker \varphi_x = \text{Span} \{v_1, \ldots, v_r\} \)

Proof that \( 2 \implies \operatorname{Im} F \) closed:

Have \( \Lambda^r V \to \operatorname{Hom}_k(V, \Lambda^{r+1} V) \)

\( x \mapsto \varphi_x \)

injective \& linear (check).

\( \implies \rk \leq n. \)

So can apply \( \operatorname{TP} \ldots \)

\( H : \operatorname{TP}(\Lambda^r V) \xrightarrow{\text{linear}} \operatorname{TP}(\operatorname{Hom}_k(V, \Lambda^{r+1} V)) \)

\( \upharpoonright F \)

\( \operatorname{Gr}, n \)

Image of \( \operatorname{Gr}, n \) lies in set \( W \) of maps of rank \( \leq n-r \). (alg cond)

\( \operatorname{Gr}, n = Z(\Z( H^* (\text{minor conditions}))) \)

\( = H^{-1}(W \cap \operatorname{Im} H) \)
Lemma. \( x \in \Lambda^r V, \ x \neq 0. \)

Then

1. \( \dim \ker \phi_x \leq r. \)
2. equality \( \iff X \) tot. dec
3. If \( x = a \cdot v_1 \wedge \ldots \wedge v_r \) tot dec
   \( \ker \phi_x = \text{Span} \{v_1, \ldots, v_r\} \)

Pf. Choose basis \( e_1, \ldots, e_n \) for \( V \)
   \( \rightsquigarrow \) basis \( e_i \) for \( \Lambda^r V \)

Assume \( e_1, \ldots, e_s \) is basis for \( \ker \phi_x \)

Pf of 1. Want \( s \leq r \)

Say \( x = \sum a_i e_i \)

Fix some \( i \in \{1, \ldots, s\} \)

\( \phi_x(e_i) = 0 \iff a_i = 0 \) when \( i \notin I \)

i.e. every non-0 term of \( x \) has an \( e_i \).

Since true for \( i \in \{1, \ldots, s\} \)

every nonzero term uses \( e_1, \ldots, e_s \)

\( \Rightarrow s \leq r \)

Pf of 2. Suppose \( s = r \).

Then \( x \) is a mult. of \( e_1, \ldots, e_s \)

other dir: \( x = v_1 \wedge \ldots \wedge v_r \) apply 0

Pf of 3. \( \text{Span} \{v_1, \ldots, v_r\} \subseteq \ker \phi_x \)

but dim's same by 2. \( \Box \)

Fact. \( x \wedge x = 0 \iff x \) tot dec.
Local coords on Grassmannian

Consider chart on $\text{Im } F$ where $a_j \neq 0$. WLOG $a_J = a_{1 \ldots r}$ (others differ by permuting coords).

Let $B = r \times n$ matrix of rank $r$ ($\text{row } B = \text{pt } \in \text{Gr}, n$)

$F(\text{row } B)$ is

$(b_{1e_1} \ldots b_{nen}) a \ldots a$

$(b_{1e_1} \ldots b_{nen}) = \sum a_j e_j$

Only the $b_{j-e_j}$ with $j \notin J$ contribute to $a_J$

Further: $a_J$ is the leftmost minor ($J = 1 \ldots r$)

\[ J = 1, 2 \]

\[ B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \end{pmatrix} \]

\[ b_{11}b_{22} e_{1} e_{2} + b_{12}b_{21} e_{2} e_{1} + \ldots \]

\[ = (b_{11}b_{22} - b_{12}b_{21}) e_{1} e_{2} + \ldots \]

So $a_J \neq 0 \iff \text{leftmost } r \times r \text{ matrix is invertible.}$

\[ \implies \text{can mult. } B \text{ on left to get} \]

\[ \begin{pmatrix} I_r & b_{1m} & \ldots & b_{1n} \\ b_{r1} & \ldots & b_{rn} \end{pmatrix} \]

bij copy of $A^{r(n-r)}$
It's a bijection since RREF unique.

It's also $\cong$ of aff. alg vars.

$\Rightarrow$ The $a_i$ are minors.

$\Leftarrow$ Need to get bij as polys in $a_i$

One example

$$a_{23\ldots r_j} = \left| \begin{array}{ccc} 0 & 0 & \cdots & 0 & b_{1j} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & b_{rj} \end{array} \right| = (-1)^{r+1} b_{1j}$$

other cases similar.

The incidence correspondence

$$I = \{ (W, V) : W \in \text{Gr}_n, \, V \in \text{P}(W) \}$$

$\subseteq \text{Gr}_n \times \text{P}^{n-1}$

Thm. $I$ is a proj subvar of

**Applications**

$\Rightarrow \bigcup_{W \in V} W \subseteq \text{P}^{n-1}$ is a subvar.

Pf idea: $I$

$$\begin{array}{ccc} \pi_1 & \Rightarrow & \pi_2 \\ \text{Gr}_n & \text{P}^{n-1} & \pi_2 \circ \pi_1^{-1}(V) \\ \bigcup_{W \in V} W \end{array}$$
② $X \subseteq \mathbb{P}^n$ par.

$L_r(X) = \text{locus of proj } r\text{-planes meeting } X.$

**Prop.** $L_r(X)$ is a proj subvar of $\text{Gr}^{r+1,n+1}$, hence $\rightarrow$

par in $\mathbb{P}^n$ by prev appl.

**Pr.**

$I$

$\pi_1 \leftarrow \pi_2 \rightarrow$

$L_r(X) = \pi_1 \circ \pi_2^{-1}(X)$
Incidence Correspondence

\[ I = I_{r,n} = \{(W, v) : v \in \mathbb{P}(W)\} \]
\[ \subseteq Gr, n \times \mathbb{P}^{n-1} \]
\[ \mathbb{P}(W) = \{ W \setminus 0 \} / \text{scale} \subseteq \mathbb{P}^{n-1} \]

Thm. \( I_{r,n} \) is proj subvar of
\[ Gr, n \times \mathbb{P}^{n-1} \]

Fact 1. \( X, Y \) proj av's
\[ U \subseteq X \text{ open } \Rightarrow U \times Y \text{ open in } X \times Y \]

Pf. Suffices closed \( x \) is closed.
\[ \Rightarrow \text{ homog. zero set of polys in } X_i \]
Can use same polys as fn's on \( X \times Y \).
Recall from Segre: Closed sets in \( X \times Y \)
are van sets of bihomog. poly's

(Topology)

Fact 2. \( A \subseteq X \) closed
\[ \iff \exists \text{ open cover } \{X_i\} \text{ of } X \text{ s.t. } X_i \cap A \text{ closed in } X_i \text{ (in subsp top)} \]

Pf \[ \iff X_i \setminus A \text{ open in } X_i \text{ hence } X \]
& \[ X \setminus A \text{ is union of these.} \]
Fact 3 (LinAlg)

\[ A = (I_r | B)^{\text{r} \times \text{n}}. \]

Then \( v \in \text{Row} A \iff \forall i \left( (i^{\text{th}} \text{col of } A) \cdot \left( \frac{v_1}{v_r} \right) = v_i \right) \)

\[ \begin{align*}
\text{PF.} & \quad v \in \text{Row} A \\
\iff & \quad v \in \text{Col} \left( \begin{bmatrix} I_r \\ B^T \end{bmatrix} \right) \\
\iff & \quad \left( \begin{bmatrix} I_r \\ B^T \end{bmatrix} \right)^T x = v \quad \text{consistent} \\
\implies & \quad \left( I_r \right)^T \left( \frac{v_1}{v_r} \right) = v \quad \square
\end{align*} \]

Fact 4 \( f \in k[x_1, \ldots, x_n, y_0, \ldots, y_m] \)

homog in y's then \( Z(f) \) is closed in \( \mathbb{A}^n \times \mathbb{P}^m \) in subspace topology. \( \square \)

PF of Thm. Cover \( Gr_n \) by open sets \( U_{a_i, i_r} \). By Facts 1+2 suffices to show

\[ (U_{a_i, i_r} \times \mathbb{P}^{n-1}) \cap I_r \text{ closed in } U_{a_i, i_r} \times \mathbb{P}^{n-1} \]

We'll do \( U_{a_i, i_r} \) i.e. subset of \( Gr_n \) given by \( \{ A : (I_r | B) \} \)

By Fact 3, the intersection given by

\[ (i^{\text{th}} \text{col of } B) \cdot \left( \frac{v_1}{v_r} \right) = v_{i+r} \]

This poly is homog in \( v_i \) closed subset of \( U_{a_i, i_r} \times \mathbb{P}^{n-1} \) \( \square \)
More variety!

Four constructions

1. **Prop.** $V \subseteq Gr, n$ subvar
   $$\Rightarrow X = U \cup W \text{ subvar of } \mathbb{P}^{n-1}$$

   **Pf.** $Ir, n$
   $$\begin{array}{c}
   \pi_1 \\
   V \subseteq Gr, n \\
   \pi_2 \\
   \mathbb{P}^{n-1}
   \end{array}$$
   $$X = \pi_2 \circ \pi_1^{-1}(V)$$

   Need: $\pi_i$ are continuous
   $\pi_i$ are closed.

2. **Prop.** $X \subseteq TP^n$ pav.
   $$L_r(X) = \text{locus of proj } r\text{-planes meeting } X$$
   **Prop.** $L_r(X) \text{ subvar of } Gr_{r+n+1} = Gr, n$
   **Pf.** $L_r(X) = \pi_1 \circ \pi_2^{-1}(X)$

3. **Joins**
   $$X, Y \subseteq TP^n \text{ subvars}$$
   $$J(X, Y) = \{ \text{lines in } TP^n \text{ meeting both} \}$$
   **Prop.** $J(X, Y) \text{ subvar of } G_{2, n+1} = G_{1, n}$
   **Pf.** $J(X, Y) = L_1(X) \cap L_2(Y)$

4. **Fano varieties**
   $$Fr(X) = \{ \text{r-planars contained in } X \}$$
   $$\subseteq Gr, n.$$
Projections are Morphisms

\[ X \in \mathbb{P}^n, \ Y \in \mathbb{P}^m \]

Segre \[ \times Y \subseteq \mathbb{P}^n \times \mathbb{P}^m \]

\[ \pi_Y : X \times Y \rightarrow Y \]

\[ (x, y) \rightarrow y \]

Prop. \( \pi_Y \) is a morphism

Prf. Suffices to do

\[ \mathbb{P}^n \times \mathbb{P}^m \rightarrow \mathbb{P}^m \]

Recall Segre:

\[ (x, y) \rightarrow \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix} (y_0 \ldots y_m) \]

\[ = \begin{pmatrix} x_0 y_0 & \ldots & x_0 y_m \\ \vdots & \ddots & \vdots \\ x_n y_0 & \ldots & y_n y_m \end{pmatrix} \]

any nonzero row is the proj to \( Y \).

(need to check agreement on overlap, but all rows are multiples so \( \checkmark \))
Prop. $\pi_Y$ is closed.

Note: false in affine case

(\text{project } x_0 y = 1 \text{ to } A',
\text{ get } A' \setminus 0)

Thm. $f : X \to Y$ morphism

\[ Z \subseteq X \text{ subvar. Then } f(Z) \subseteq Y \text{ subvar. (all proj)} \]

Cor. $X$ connected proj var

Then any (global) regular $f_n$ is const.

If. $X \xrightarrow{\text{reg } f_n} \mathbb{A}^n \hookrightarrow \mathbb{P}^1$ not surj

image is a subvar by Thm

$\Rightarrow$ image is finite set of pts. Done by connected. $\square$

Tool: Graphs

$f : X \to Y$ morphism

$\leadsto f_f : X \to X \times Y$

$x \mapsto (x, f(x))$

Image $\Gamma_f$ is graph of $f$.

Lemma. $\Gamma_f$ closed in $X \times Y$

& $f_f : X \to \Gamma_f$ is $\subseteq$.

To prove Thm. Lemma allows us to assume $f$ is $\mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^n$
Projections are closed
\[ X \subseteq \mathbb{P}^n, \quad Y \subseteq \mathbb{P}^m \]
\[ \leadsto X \times Y \subseteq \mathbb{P}^n \times \mathbb{P}^m \]
\[ \pi_Y : X \times Y \rightarrow Y \]

Prop. \( \pi_Y \) closed.

(\text{false for affine! hyperbola!})

More generally...

Thm. Any \( f : X \rightarrow Y \) morphism
is closed

“compactness property”

Cor. Global reg \( f \) const.
if \( X \) conn.

Graphs \( f : X \rightarrow Y \)
\[ \iota_f : X \rightarrow X \times Y \]
\[ x \mapsto (x, f(x)) \]
\[ \text{Image}(\iota_f) = \pi_f \]
\[ \pi_f = \{ (x, y) \in \mathbb{P}^n \times \mathbb{P}^m : f(x_0, \ldots, x_n) = y_i \forall i \} \]
assuming \( f = (f_0, \ldots, f_m) \) on open set in \( X \)

Prop1. \( \pi_f \) closed in \( X \times Y \)
\[ \iota_f : X \rightarrow \pi_f \text{ is } \subseteq. \]
Prop 1. \( \Gamma_f \) closed in \( X \times Y \)
\( f: X \to \Gamma_f \) is \( \cong \).

**Morphism.** At any \( x \in X \)

Find open \( U \) s.t. \( f \) given by \( f_0, \ldots, f_m \in k[x_0, \ldots, x_n] \)

same deg. On \( U \), post-comp with Segre map gives \( \prod i \neq j \)

\( (x_0) (f_0 \ldots f_m) \to x_i f_j \)

- this agrees on overlaps
- same deg
- image in \( \Gamma_f \)
- \( y_i f_j \) don't sim. vanish

Closed. Let \( (p, q) \notin \Gamma_f \) i.e. \( f(p) \neq q \)

Choose \( U \subseteq P^n \) nbd of \( p \) so \( f \) def on \( U \)

by \( f_0, \ldots, f_m \) of deg \( d \).

Let \( Z \subseteq P^n \times P^m \) van set of \( 2 \times 2 \) minors of \( (f_0 \ldots f_m) \) e.g. \( f_0 y_1 = f_1 y_0 \)

- bihomog. of deg \( (d, 1) \)
- \((U \times P^m) \cap Z^c \) open nbd of \( (p, q) \) in \( \Gamma_f^c \)

exactly

(\( \Gamma_f \) would be \( \Lambda Z \). Problem is that \( f \) is only def. locally.)

Isomorphism inverse is projection \( \square \)
Prop. \( \pi : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m \)

"Main thm of elimination theory"

Lemma. \( g_1, \ldots, g_r \in \mathbb{P} \left( k[x_0, \ldots, x_n] \right) \)

\( \deg d \)

Regard \( g_i \in \mathbb{P}^N \) (take coeffs)

Let \( D = d \). Then

\[ \left\{ (g_1, \ldots, g_r) \in \mathbb{P}^N \right\}^C : \]

\[ k[x_0, \ldots, x_n]_D \subseteq (g_1, \ldots, g_r)_D \]

\[ \text{elts of} \quad \deg D \]

in the ideal

\[ N = \binom{n+d}{d} \]

\[ \text{is closed in} \quad \mathbb{P}^N \]

\[ \text{open} \]

Gathmann

**Pf of Lemma** The condition

\[ k[x_0, \ldots, x_n]_D \subseteq (g_1, \ldots, g_r)_D \]

equiv to

\[ k[x_0, \ldots, x_n]_D = (g_1, \ldots, g_r)_D \]

Since \( (g_1, \ldots, g_r) = \left\{ \sum h_i g_i : h_i \in k[x_0, \ldots, x_n] \right\} \)

\( \ast \) equiv to:

\[ F_D : (k[x_0, \ldots, x_n]_{D-d})^r \to k[x_0, \ldots, x_n]_D \]

\[ (h_1, \ldots, h_r) \mapsto \sum h_i g_i \]

being surjective, ie has

\[ \text{rank dim } k[x_0, \ldots, x_n]_D = \binom{n+d}{D} \]

\[ \iff \text{one of the minors of } F_D \]

of that dim is not zero.

\( \square \)
Prop. \( \varpi : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m \)

closed.

If. Take coords \( x_0, \ldots, x_n \), \( y_0, \ldots, y_m \).

Let \( Z \subseteq \mathbb{P}^n \times \mathbb{P}^m \).

Say \( Z = Z(f_1, \ldots, f_r) \).

\( f_i \) of deg \((d, d)\).

Let \( a \in \mathbb{P}^m \).

Let \( g_i = f_i(\cdot, a) \).

\( g_i \in k[x_0, \ldots, x_n] \).

Will show \( a \notin \varpi(Z) \) open condition.

\[
\forall \ x \in \mathbb{P}^n \text{ s.t. } (x, a) \notin Z \iff Z_\mathbb{P}(g_1, \ldots, g_r) = \emptyset \iff \sqrt{(g_1, \ldots, g_r)} \not\subseteq (x_0, \ldots, x_n) \iff \exists d_i \text{ s.t. } x_i \notin (g_1, \ldots, g_r) \forall i \iff k[x_0, \ldots, x_n]_D \subseteq (g_1, \ldots, g_r)_D \text{ some } D \text{ take } D = \Sigma d_i \text{ open condition on coeffs of } g_i \text{ (lemma)}.
\]

The coeffs of \( g_i \) are poly's in \( a \), i.e. coords on \( \mathbb{P}^m \).
Thm. Any $f: X \to Y$ morphism is closed

Proof. Say $Z \subseteq X$ closed.

$J_f: X \xrightarrow{\sim} \Gamma_f$ (Prop 1)

$\Rightarrow J_f(Z)$ closed in $\Gamma_f$,
hence in $\mathbb{P}^n \times \mathbb{P}^m$

By Prop 2 $\pi(J_f(Z)) = f(X)$
closed in $\mathbb{P}^m$

It is contained in $Y$, hence closed in $Y$ \qed

Chap 4 Dim, deg, smoothness.

$V = \text{vect sp.}$

$\dim V = \sup \{ r : \exists \text{ strictly dec chain of lin subsp} \}

\quad V = V_0 \supset V_1 \supset \cdots \supset V_r \}$

$X = \text{top space}$

Krull dimension is

$\dim X = \sup \{ r : \exists \text{ strictly dec chain of closed irred sets} \}

\quad X = X_0 \supset \cdots \supset X_r \}$

Example. $\dim \mathbb{A}^1 = \dim \mathbb{P}^1 = 1$
Chap 4. Dim, deg, smoothness.

$V = \text{vect sp.}$

$\dim V = \sup \{ r : \exists \text{ strict dec chain of subsp.} \}$

$X = \text{top sp. Krull dim}$

$\dim X = \sup \{ r : \exists \text{ strict dec chain of closed irreducible subsp.} \}$

$X = X_0 \supseteq \cdots \supseteq X_r (\neq \emptyset)$

& $\dim \emptyset = \infty.$

$X = \text{variety}$

$\dim X = \text{Krull dim in Zar. top.}$

Example. $\dim \mathbb{A}^1 = \dim \mathbb{P}^1 = 1$

Facts

1. If $X \neq \emptyset$, Hausdorff then $\dim X = 0$

(Hausdorff $\Rightarrow$ only irreducible are points)

2. $\dim X = \sup \{ \dim X_i : X_i \text{ irreducible} \}$

& strict if no irreducible component of (closure of) $Y$ is irreducible component of $X$.

3. $Y \subseteq X \Rightarrow \dim Y \leq \dim X$

4. $X$ covered by $U_i$ open

$\implies \dim X = \sup \dim U_i$
Cor. of 3: $X$ irreducible, $\dim X = 0$ 
\[ \Rightarrow X = \text{pt.} \]

Want: $\dim A^n = n$.
\[ \text{easy: } \geq n. \]

Krull dim

$A$ = ring 
$\dim A = \sup \{ r : \exists \text{ strict inc. } P_0 \subset \cdots \subset P_r \}$

proper prime ideals

By our dictionary: $\dim X = \dim k[X]$.

Prop. $\dim k[x_1, \ldots, x_n] = n$

Cor. $\dim A^n = n$

Cor. $\dim \mathbb{P}^n = n$ by 4

Example. $\dim \text{Gr}, n = r(n-r)$

(by)

\[ (I \mid \quad) \quad r \times (n-r) \]

also using 4:

$\ln k[x,y]$: $0 \subset (x) \subset (x,y) \subset k[x,y]$
Prop. \text{dim } k[x_1, \ldots, x_n] = n

If. Induct on $n$.

$n = 0 \checkmark$

Inductive step

Say:

$O = P_0 \subset P_1 \subset \cdots \subset P_m \subset k[x_1, \ldots, x_n]$

WLOG: $P_i = (f)$ where $f$ monic in $x_n$

↑ can assume $P_i$ principal

Since $k[x_1, \ldots, x_n]$ UFD.

In a non-UFD (prime) might not be prime.

Monic in $x_n$: leading term $x_n^d$

In quotient $k[x_1, \ldots, x_n]/p$,

Show

$O = \overline{P}_1 \subset \cdots \subset \overline{P}_m$ is str. inc.

chain of prime ideals.

Now use:

$k[x_1, \ldots, x_{n-1}] \rightarrow k[x_1, \ldots, x_n]/P_i$

$x_i \mapsto \overline{x}_i$

pull back $\overline{P}_i$. Get chain of prime ideals in

$k[x_1, \ldots, x_{n-1}]$

Why is preim of $\overline{P}_2$ not 0?
Example

\[ A = \frac{k[x,y]}{(y^2 - x^3 + x)} \]

\( P = \) prime in \( A \)

\( \varphi: k[x] \rightarrow A \)

\[ x \mapsto \overline{x} \]

Want \( \varphi^{-1}(P) \neq 0 \).

Subexample. Why is \( \varphi^{-1}(y) \neq 0 \)?

\[ x - x^3 \mapsto y^2 \in \langle y \rangle. \]

Next example

\[ A = \frac{k[x,y]}{(y^2 - x^3 + xy)} \]

Want \( \varphi^{-1}(y) \neq 0 \).

\[ f \in k[x][y], \quad \varphi(\text{const-in-y term}) \in \langle y \rangle \]

\[ y^2 + (x)y - x^3 \]

\( \varphi(x^3) \in \langle y \rangle \)

Where using monic??
Next goal
$X \subseteq \mathbb{P}^n$ variety

$\dim X = \frac{\text{the } d \text{ s.t.}}{
\exists \text{ finite map } X \to \mathbb{P}^d
}$

Finite maps
Defn 1. $f : X \to Y$ with dense image
and s.t. $f_\ast : k[Y] \to k[X]$ finite,
meaning $k[X]$ f.g. module over $\text{im } f_\ast$

Defn 2. $f : X \to Y$ dense image
& pt preimages are finite.

Why does every variety have
a map to $\mathbb{P}^d$ with finite
pt preimages?

Geom answer:

Stereographic proj $X \to \mathbb{P}^{n-1}$
with finite pt preims:
pt preims are $\mathbb{P}^n \cap X = \text{finite}$
Can iterate until get surj. map to $\mathbb{P}^d$
Noether normalization

Thm. \( A = \text{fin gen } k\text{-alg} \Rightarrow \exists y_1, \ldots, y_d \in A \text{ indep. st. } A \text{ is } fg \text{ as } k[y_1, \ldots, y_d] \text{ module.} \)

On last slide \( A = k[X] \).

(\text{Can deduce Nullstellensatz from this.})

Think of \( y \)'s as transcendental/indep and rest of \( A \) as dep. on those.

Example. \( A = k[x_1, x_2]/(x_2^2 - x_1^3 + x_1) \) (as above)

\( d=1 \), \( y_1 = x_1 \), \( x_2 \) satisfies \( f \in k[x_1][z] \)

\( f(z) = z^2 - (x_1^3 - x_1) \)

\( \sim \) \( A = \{ k[x_1] + x_2 k[x_1] \} \)

i.e. \( A \) gen by \( x_2, 1 \) as \( k[x_1] \text{ module.} \)

Notice \( F \) is monic in \( z \). Can always do lin. change of coords to make it so. The pf follows then as in example.
PF of NN in special case:

A gen by one elt \( c \).

(as \( k \)-mod)

If \( c \) transc. \( \Rightarrow A = k[c] \) done.

If \( c \) alg \( \Rightarrow f(c) = 0 \) \( f \) monic deg \( d \)

\( \Rightarrow A = k[z]/(f(z)) \)

\& \( A \) gen as a module

by \( 1, c, \ldots, c^{d-1} \)
Last time:
\[ \dim X = \dim k[X] = d \text{ s.t. } \exists \text{ finite } X \to \mathbb{P}^d. \]

Today:
\[ \dim X = \min_{p \in X} \dim T_p X = \text{tr } \deg_k k(X) \]

**Tangent spaces**
\[ X = \mathbb{Z}(f) \subseteq \mathbb{A}^n \text{ hypersurf.} \]
\[ \nabla f_p = \left( \frac{df}{dx_1}(p), \ldots, \frac{df}{dx_n}(p) \right) \in \mathbb{k}^n. \]
\[ \nabla f_p \in (\mathbb{k}^n)^* \text{ via dot product.} \]
\[ T_p X = p + \ker \nabla f_p \]

Write
\[ f_p^{(1)} = \sum \left( \frac{df}{dx_i}(p) \right)(x_i - p_i) \]

"Linear part of \( f \) at \( p \)"

\( T_p X \) is the set of solns.
Examples

6) Hyperplane $H \subseteq \mathbb{A}^n$
   $T_pH = H$ (exercise).

1) Parabola $f(x,y) = y - x^2$
   $\nabla f = (-2x, 1)$
   $\nabla f_0 = (0, 1)$
   $\sim T_0X = \mathbb{A}^2$

2) $X = \mathbb{Z}(y^2 - x^2 - x^3)$
   $\sim T_0X = \mathbb{A}^2$

3) $X = \mathbb{Z}(y^2 - x^3)$

4) $X = \mathbb{Z}(x^{m} - y^{m})$
   Check: $T_{(a,b)}X$ is a line if char $k \nmid m$.  
   not irreducible - so we need more definitions!
Projective varieties

To define $T_p X$, pass to affine chart, take tangent space there, take projection closure.

Tangent spaces & roots

Prop. $L \subseteq \mathbb{A}^n$ affine line, $p \in L$;

$X = \mathbb{Z}(f) \subseteq \mathbb{A}^n$.

Then $L \subseteq T_p X \iff f|_L$ has a multiple root at $p$.

Examples.
1. $X = \mathbb{Z}(y-x^2),
   \quad x^2 = 0$.
2. $X = \mathbb{Z}(y^2 - x^3).
   \quad L : y = tx
   \quad (tx)^2 - x^3 = x^2(t^2 - x)$
   mult. root at $0$.

Prop. Let $L(t) = (p_1 + b_1 t, \ldots, p_n + b_n t)$

Let $g(t) = f|_L = f(p_1 + b_1 t, \ldots, p_n + b_n t)$

Know $g(0) = f(p) = 0$.
Want $g'(0) = 0$. By chain rule:

$\frac{dg}{dt}(0) = 0 \iff \sum b_i \frac{df}{dx_i} (p) = 0 
\iff L \subseteq T_p V$. 

\[ \blacksquare \]
Tangent sq for general irred (not just hypersurf)

\[ X \subseteq \mathbb{A}^n \quad X = \mathbb{Z}(f_1, \ldots, f_m) \]

\[ T_p X = \bigcap_{f \in I(X)} T_p \mathbb{Z}(f) \]

exercise \[ = \bigcap_{i=1}^m T_p \mathbb{Z}(f_i) \]

Smoothness

\[ X = \mathbb{Z}(f) \subseteq \mathbb{A}^n \quad \text{irred hypersurf} \]

\[ p \in X \quad \text{smooth if } \nabla f_p \neq 0 \quad \iff T_p X = \mathbb{A}^{n-1} \]

\[ \text{singular o.w.} \quad \iff T_p X = \mathbb{A}^n \]

\[ \implies X_{\text{smooth}} \quad X_{\text{sing}} = X \setminus X_{\text{smooth}}. \]

Examples
Prop. \( X = \mathbb{Z}(f) \subset \mathbb{A}^n \) irred.
\( X_{\text{smooth}} \subseteq X \) open, dense

**PF (char \( k = 0 \))**

To show: 1. \( X_{\text{sing}} \) closed
2. \( X_{\text{smooth}} \neq \emptyset \).

1. \( X_{\text{sing}} = \mathbb{Z}(f, \frac{df}{dx_1}, \ldots, \frac{df}{dx_n}) \)

2. Assume \( X_{\text{sing}} = X \).
\( \Rightarrow \frac{df}{dx_i} \in (f) \quad \forall i \) since \( f \) irred.

Since \( f \) not const, this is a contrad. (look at degrees)

Note over \( \mathbb{C} \), \( X_{\text{smooth}} \) is a complex manifold (inverse fn thm).

The reducible case

If \( X \) has irred. comp's \( \{X_i\} \), say \( p \) is smooth if it lies in exactly one \( X_i \) & is smooth as a pt in \( X_i \).

\( \overline{[1]} \)

Since \( f \) not const, this is a contrad. (look at degrees)

\( X = \mathbb{Z}(xy, xz) \)

\( O \) is smooth in both components, but not smooth by our defn.
Back to dimension

Let's write
\[ \dim X = \min_{p \in X} \dim T_p X \]
for \( X \) irreductible.

If \( X \) is reducible with irreductible components \( X_i \),
\[ \dim X = \max \dim X_i . \]

Above examples \( 1 - 4 \) have \( \dim 1 \).

Prop. \( X = \mathbb{Z}(f) \not\subseteq \mathbb{A}^n \) hypersurf.
\[ \Rightarrow \dim X = n - 1 . \]

Prop. \( X \subseteq \mathbb{A}^n \) irreductible.

\[ \exists \text{ open, dense } X_0 \subseteq X \text{ s.t.} \]
\[ \dim T_p X = \dim X \quad \forall p \in X_0 . \]

Lemma. \( X \subseteq \mathbb{A}^n \) irreductible. \( \forall r \in \mathbb{N} \). The set
\[ S_r(X) = \{ p \in X : \dim T_p X \geq r \} \] is closed.

If \( \text{Say } \Phi(X) = (f_1, \ldots, f_m) \)
\[ T_p X = \bigcap \mathbb{Z}(f_i)^{(i)}_p \]
\[ \Rightarrow \dim T_p X = n - \text{rank} \left( \frac{df_i}{dx_j}(p) \right) \]

Pf of Prop. Let \( r = \dim X \leadsto S_r X = X , S_{r+1} X \neq X . \]
Back to smoothness

$X$ irred, maybe not hypersurf.

$X = \mathcal{Z}(f_1, \ldots, f_m)$

$p \in X$ is smooth if

$\text{rank} \left( \frac{\partial f_i}{\partial x_j} \right) = m.$

Fact.

$p \text{ smooth } \iff \text{dim } T_p X = \text{dim } X$

Codim.

$X = \mathcal{Z}(f_1, \ldots, f_m) \text{ irred. } \subseteq \mathbb{A}^n$

$\text{codim } X = n - \text{dim } X$

$= \text{rank} \left( \frac{\partial f_i}{\partial x_j} (p) \right) \quad p \text{ smooth}$

$\Rightarrow \text{codim } X \leq m.$

(also true for red.)
We'll show
\[ \dim X = \text{trdeg}_K k(X) = \dim k[X] \]

Hypersurface case
\[ k[X] = k[x_1, \ldots, x_n]/(f) \]

WLOG \( f \) uses \( x_1 \)
\[ k(X) = k(x_2, \ldots, x_n)[x_1]/(f) \]

which clearly (??) has transc. deg \( n-1 \).
Goal for today:
- Coord free description of tangent spaces.

We'll show

\[ T_p X \cong (M/M^2)^* = (m/m^2)^* \]

\[ M \leq k[x_1, \ldots, x_n] \]

\[ m \leq k[x_1, \ldots, x_n] \]

max ideals at \( p \):

\[ M = (x_1 - p_1, \ldots, x_n - p_n) \]

fns that vanish at \( p \)

---

**Cotangent spaces**

\[ T_p^* V = \text{dual of } T_p V \]

\[ = \{ \text{linear } T_p V \to k \} \]

\[ = \text{"linear forms"} \]

**Notation:** \( g \in k[x_1, \ldots, x_n] \)

\[ dg = \text{diff. of } g \text{ at } p. \]

e.g. \( g(x,y) = x^2 + xy + x \)

\[ dg = (2x + y + 1) \frac{d}{dx} + x \frac{d}{dy} \]

\[ dg = \frac{d}{dx} \in (k^n)^* \]
Prop. Let $X = \mathbb{Z}(f_1, \ldots, f_m) \subseteq \mathbb{A}^n$

$p \in X \quad g \in k[X]$

$\Rightarrow dg$ is lin form on $T_pX$.

PF. To show well-def.

Say $G_1, G_2 \in k[x_1, \ldots, x_n]$ rep. $g$.

$G_1 - G_2 = \sum h_i \cdot f_i \cdot h_i \in I(X)$

$\Rightarrow d_p(G_1 - G_2) = \sum (d_p h_i)(f_i(p) \cdot \text{product rule} + h_i(p)(d_p f_i))$

O by defn

$= 0 \quad \square$

Prop. Same $X$. Differentiation induces surj $M \rightarrow T_p^*V$

with kernel $M^2$.

PF. Setup. WLOG $p = 0$.

WLOG $T_pV = \langle x_1, \ldots, x_r \rangle$ (change of coords)

Let $\tilde{M} = (x_1, \ldots, x_n) \subseteq k[x_1, \ldots, x_n]$

Its image in $k[X]$ is $M$. 

$T_pV$ defined so that this form evals to 0 on it.
**Prop.** Same $X$. Differentiation induces surj $M \rightarrow T_p^*X$ with kernel $M^2$. So: $T_p^*X = M/M^2$.

**Pf.** Setup. WLOG \( p = 0 \).

WLOG $T_pX = \langle x_1, \ldots, x_r \rangle$ (change of coords).

Let $\tilde{M} = (x_1, \ldots, x_n)$.

Its image in $k[X]$ is $M$.

Surjectivity: let $l = \sum c_i x_i \in T_p^*X$.

Then $L = \sum c_i x_i$ has $dL = l$.

Kernel: Say $g \in M$, $d_0 g = 0 \in T_p^*X$ & $g$ is image of $G \in \tilde{M}$.

So $d_0 G = 0$ on $T_0 X$ (first Prop).

Then $d_0 G = \sum x_j (d_0 f_j)$ (by defn of $T_pX$).

Let $\tilde{G} = G - \sum x_j f_j$.

Then $\tilde{G}$ still maps to $g$ in $k[X]$.

But $d_0 \tilde{G} = 0$ on $T_0 \tilde{M}$.

$\Rightarrow$ const & lin. terms of $\tilde{G}$ vanish.

$\Rightarrow \tilde{G} \in \tilde{M}^2 \Rightarrow g \in M^2$. \(\Box\)
Moving the \*$

\[ R = \text{ring}, \ M \subseteq R \text{ max ideal} \]
\[ \sim R \cdot M \leq M, \ R \cdot M^2 \leq M^2 \]
So \( M, M/M^2 \) modules over \( R \)
Also, mult by \( M \) on \( M/M^2 \) is 0 map.
So \( M/M^2 \) is \( R/M \)-module
i.e. \( M/M^2 \) is vect sp. over \( R/M \)
So \( T_p V = (M/M^2)^* \) makes sense

\((M/M^2)^*\) is called Zariski tangent sp.

Differentials

\[ \text{Prop. } f : X \rightarrow Y \text{ morphism of aa's} \]
\[ \sim f_* : T_p X \rightarrow T_{f(p)} Y \]
\[ \text{If. } f_* : k[Y] \rightarrow k[X] \]
\[ \text{preim of } M \text{ is } N = \text{max ideal for } f(p) \]
\[ = \text{fns vanish at } f(p) \]
So \( N/N^2 \rightarrow M/M^2 \)
Coord free descr. of differential

**Prop.** $X \leq \mathbb{A}^n$ irredu.

$f \in k[X]

Then $f - f(p) \in M$.

and df = image of $f - f(p)$

in $M/M^2 = T^* p V$

**Proof.** Subtracting $f(p)$ kills const term.

Modding by $M^2$ kills quad & higher terms.

you!

---

**Example**

$X = \mathbb{Z}(x^3 - y^2) \leq \mathbb{A}^2$

At $p = (1,1)$ can see dim $M/M^2 = 1$:

$M = (x-1, y-1)

\rightarrow M^2 = (x^2 - 2x + 1, (x-1)(y-1), y^2 - 2y + 1)

\rightarrow y - 1 = (x^3 + 1)/2 - 1

= (x(2x-1)+1)/2 - 1

= (2x^2 - x + 1)/2 - 1

= (3x - 1)/2 - 1

= 3/2 (x - 1)

At $p = (0,0)$ can see dim $M/M^2 = 2$:

$M = (x,y)$, $M^2 = (x^2, xy, y^2)$

$\rightarrow M/M^2 = \{ ax + by \}$
Projective varieties
\[ \mathcal{O}_X^p = \{ f/g \in K(X) : g(p) \neq 0 \} \]
\[ m \in \mathcal{O}_X^p \text{ s.t. } f(p) = 0 \]
max ideal.

Lemma. \( X, M, m, p \) as above.
\[ M/M^2 \cong m/m^2 \]

Pf. WLOG \( p = 0 \).
Inclusion \( M \hookrightarrow m \)
Induces injection
\[ M/M^2 \hookrightarrow m/m^2 \]

Surj. Let \( f/g \in m/m^2 \) so \( g(0) \neq 0 \)
\[ \sim f/g(0) - f/g \]
\[ = f \left( \frac{1}{g(0)} - \frac{1}{g} \right) \in m^2 \]
So \( f/g(0) = f/g \) in \( m/m^2 \)

in \( M \subseteq k[X] \)

Cor 1. \( f : X \rightarrow Y \) rat.
\[ \sim f_* : T_p X \rightarrow T_{f(p)} Y. \]

Cor 2. \( X, Y \) birat \( \Rightarrow \dim X = \dim Y \)
Back to dim

Thm. \( X \subseteq \mathbb{A}^n \) is irreducible

\[ \dim X = \text{trdeg}_k k(X). \]

Also: \( \text{trdeg}_k k(X) = \dim k[X] \).

If it's true for hypersurfaces, true for all \( X \) since every \( X \) is birational equivalent to a hypersurface. (Noether normalization).

For hypersurfaces:

We proved \( \dim = n - 1 \) so suff. to show \( \text{trdeg} = n - 1 \)

\[ X = \mathbb{Z}(f) \subseteq \mathbb{A}^n \] if irreducible.

\[ k[X] = k[x_1, \ldots, x_n]/(f) \]

WLOG \( f \) uses \( x_1 \)

\[ k(X) = k(x_2, \ldots, x_n)[x_1]/(f) \]

transcendental basis.
Blowups

or: Zooming in

Two problems
1. Varieties have singularities
   \[ \mathcal{X} \]
2. Rational maps not def everywhere
   \[ \mathbb{P}^n \to \mathbb{P}^{n-1} \]
def. on \[ \mathbb{P}^n \backslash a \]
No way to extend over \[ a \].

Blowup is a tool for fixing these.

Idea of blowup
Replace pt \( p \) with set of lines thru \( p \)

Picture over \( \mathbb{P}^1 \):

- M"obius band
- \( \text{Id, opp pts on inner circle} \)
- \( \text{polar coords} \) (almost)
- \( \text{take preim of smooth part, then take closure singularity gone!} \)
The blowup of $\mathbb{A}^2$ at $O$

\[
\pi : \mathbb{A}^n \setminus O \to \mathbb{P}^{n-1}
\]

\[(a_1, \ldots, a_n) \mapsto [a_1 : \ldots : a_n]
\]

$\Gamma_\pi \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1}$ graph.

$\widetilde{\mathbb{A}}^2 = \text{Zar. closure of } \Gamma_\pi \text{ in } \mathbb{A}^n \times \mathbb{P}^{n-1}$.

$\widetilde{\mathbb{A}}^2$ blowup of $\mathbb{A}^2$ at $O$.

$n=2$ case

$\pi(x,y) = [x:y] \text{ (or } x/y)\)

$\widetilde{\mathbb{A}}^2 = \{(x,y), [t_0:t_1] : xt_1 = yt_0\}$

Check: this is the closure of $\Gamma_\pi$.

Projection to $\mathbb{A}^2$ induces

$p : \widetilde{\mathbb{A}}^2 \to \mathbb{A}^2$

and $p^{-1}(x,y) = \{[(x,y), [x:y]] : (x,y) \neq O\}$

Fact. $p$ induces $\widetilde{\mathbb{A}}^2 \setminus E \xrightarrow{\sim} \mathbb{A}^2 \setminus O$

$E = \text{exceptional line/divisor}$
Affine cover of $\widetilde{\mathbb{A}}^2$

$\mathbb{P}^1$ has std. aff. cover $U_0, U_1$.

$\sim \mathbb{A}^2 = V_0 \cup V_1, \quad V_i \subseteq \mathbb{A}^2 \times \mathbb{A}^1$

where

$V_0 = \{(x, y, [1: t_1]) : xt_1 = y \}$

$V_1 = \{(x, y, [t_0 : 1]) : x = y t_0 \}$

Note: $V_i \cong \mathbb{A}^2$

$V_0$ coords: $x, u = t_1$

$V_1$ coords: $y, v = t_0$

So

$V_0 = \{(x, u, x), [1: u]\} = \{(x, u)\}$

$V_1 = \{(y, y, y), [v:1]\} = \{(y, v)\}$

Under $p: \widetilde{\mathbb{A}}^2 \to \mathbb{A}^2$

Hor lines $\longrightarrow$ lines thru origin (get all but vertical)

Vert lines $\longrightarrow$ vert lines.

Similar for $V_1$. 

Resolving singularities

Say \( X \subset \mathbb{A}^n \) sing. set \( S \)

A resolution is

\[ p: \tilde{X} \rightarrow X \text{ s.t. } \tilde{X} \text{ nonsing} \]

\& restr. \( \tilde{X} \setminus p^{-1}(S) \rightarrow X \setminus S \)

is an isomorphism.

Resolution for

- curves: blow up pts
- surfaces over \( \mathbb{C} \): Jung, Walker
- Zariski '35

3-folds char \( = 0 \): Zariski

Annals '44

3-folds char \( \neq 0 \): Abhiyankar (Z's student)

All varieties char 0: Hironaka ~'70

char \( \neq 0 \) open.

We'll look at curves \( \tilde{X} \).
Example 1

\[ C = \mathbb{Z}(x^2-y^2) \]

Resolution:

Higher dim version:

\[ X = \mathbb{Z}(x^2+y^2-z^2) \]
\[ \tilde{X} = \mathbb{Z}(x^2+y^2-1) \]
\[ \tilde{X} \rightarrow X \]
\[ (x,y,z) \mapsto (x^2,y^2,z) \]

\[ x_0 \text{ plane} \rightarrow \text{pt} \]

Example 2

\[ C = \mathbb{Z}(y^2-x^2-x^3) \]
\[ p^{-1}(C) = \{(x,y), [t_0:t_1] : y^2 = x^3 + x^2, t_0y = t_1x\} \]
\[ p^{-1}(C) \cap V_0 = \{(x,xu), [1:1] : x^2(x+1-u^2)=0\} = \{(x,u) : x^2(x+1-u^2)=0\} \subseteq \mathbb{A}^2 \]
\[ p^{-1}(C) = \text{parabola \setminus pt} \]
\[ \text{closure } \tilde{C} \text{ is parabola.} \] Smooth!

Example 3

\[ C = \mathbb{Z}(y^2-x^3) \]
\[ p^{-1}(C) \cap V_0 = \{(x,u) : (xu)^3 = x^3\} \]
\[ \rightarrow \text{parabola.} \]
\[ \rightarrow \text{parabola.} \]

Aside: link of cusp is \((3,2)\)-cusp on \(T^2\) (trefoil)
Blowing up higher-dim subvars

Algebra version: 

\[ Y \subseteq X \subseteq \mathbb{A}^n \] aav's

\[ Y = Z(f_0, \ldots, f_m) \quad f_i \in k[X] \]

Define:

\[ \varphi : X \longrightarrow \mathbb{P}^m \]

\[ x \mapsto [f_0(x) : \ldots : f_m(x)] \]

regular on \( X \setminus Y \)

\[ \Gamma_\varphi \subseteq \mathbb{A}^n \times \mathbb{P}^m \quad \text{and} \quad p : \Gamma_\varphi \rightarrow X \]

closure is \( \mathcal{B}_Y(X) \) blowup of \( X \) at \( Y \).

\[ p^{-1}(Y) \quad \text{"exceptional divisor"} \]

Example:

\[ \mathcal{O} = Y \subseteq X = \mathbb{A}^2 \]

\[ Y = Z(x, y) \]

\[ \varphi : \mathbb{A}^2 \rightarrow \mathbb{P}^1 \]

\[ (x, y) \mapsto [x : y] \]

Can do similar for proj var's

(Use homog. polys).

Topological version:

Read in Harris.

Idea: replacing pts in \( Y \) with space of normal directions.

\[ \text{e.g. } Y = Z-\text{axis in } \mathbb{A}^3 \text{; pts in } Y \text{ get replaced with } \mathbb{P}^1 \]
Theorem $X$ variety

$\varphi : X \longrightarrow \mathbb{P}^n$ rational

Then $\exists$

\[
\begin{array}{c}
\text{blowups} \quad \downarrow \quad \text{morphism} \\
\quad \downarrow \\
\quad \downarrow \\
X = X_0 \longrightarrow \mathbb{P}^n
\end{array}
\]

So: a rational map is a regular map on some blowup.
Resolution of singularities of an algebraic variety over a field of characteristic 0."

Annals of Math.
Degree

\( X = Z(f) \) hypersurf.

\( \sim \) \( \deg X \) defined as \( \deg f \).

More generally:

\( X \subseteq \mathbb{P}^n \) irreducible, \( k \)-dim

\( \sim \) \( \deg X \) is:

1. deg of any hypersurf in \( \mathbb{P}^{k+1} \) birat eq to \( X \)

2. the deg of a cover \( X \to \mathbb{P}^k \)

3. \# pts of int. of generic \((n-k-1)\)-plane with \( X \)

___

If \( X \) is a complex manifold

\( \sim [X] \in H_{2k}(\mathbb{P}^n; \mathbb{Z}) \cong \mathbb{Z} \)

is the degree.

(also works for singular \( X \)).
Chapter 5 Curves or Bézout & applications.

$k = \text{alg closed.}$

B's thm. $C, D \subseteq \mathbb{P}^2$ curves of deg $m, n$. If $C, D$ have no irreducible components in common then they intersect $mn$ times with mult.

**Schematic**

Special cases

1. $C, D \text{ lines } \sim 1 \text{ pt.}$

   $\mathbb{P}^2$ exists so all lines intersect. From this: all curves intersect the right # of times.

   (Like how solving $x^2 + 1 = 0$ allows to solve all polynomials)
Example. \( C = \mathbb{Z}(yz-x^2) \)
\[ D = \mathbb{Z}(z-ax) \]
\[ C \cap D = (yz-x^2, z-ax) \]
\[ = (axy-x^2) \]
Set \( y=1 \): \( ax-x^2=0 \) ~ \( x=0,a \)
~ \([0:1:0] \) & \([a:1:a^2]\)

When \( a=0 \) get one pt of mult 2.

Special case of (2): Every conic meets line at \( \infty \) in 2 pts w/mult.

You finish the proof of Bezout in this case.
Special case of 2. Every conic meets line at $\infty$ in 2 pts w/mult.

e.g. circle $(x-a)^2 + (y-b)^2 = r^2 z^2$

always contains $[1:i:0] \& [1:-i:0]$

If C, D both circles, they meet at those 2 pts plus 2 more in $\mathbb{A}^2$

unless... concentric, in which case the 2 pts at $\infty$ have mult 2.

Similar: hyperbola meets line at $\infty$ at the asymptotes

• parabola meets it at 1 pt with mult 2. (prev. ex. a=0)

Example C = $Z(x^2 + y^2 - z^2)$

D = $Z((x-z)^2 + y^2 - z^2)$

circles

you: find the 2 pts not at $\infty$. 
**Resultants**

Goal: find common zeros of two polys

Say \( f(x) = a_0 + \ldots + a_m x^m \)
\( g(x) = b_0 + \ldots + b_n x^n \)

The resultant \( \text{Res}(f,g) \) is det of the \((m+n) \times (m+n)\) Sylvester matrix.

Prop. \( \text{Res}(f,g) = 0 \iff Z(f) \cap Z(g) \neq \emptyset \)

equiv: \( f, g \) no common factors.
Linear case
\[ a_0 + a_1 x = 0 \]
\[ b_0 + b_1 x = 0 \]
\[ \sim (a_0 \ a_1) \]
\[ (b_0 \ b_1) \]

Quadratic case
\[ a_0 + a_1 x + a_2 x^2 = 0 \]
\[ b_0 + b_1 x + b_2 x^2 = 0 \]
\[ \sim \begin{pmatrix} a_0 & a_1 & a_2 & 0 \\ 0 & a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 & 0 \\ 0 & b_0 & b_1 & b_2 \end{pmatrix} \]

Can see rank \( \geq 3 \) (look at 1st 3 rows)
\[ \Rightarrow \dim \ker \leq 1 \]
So \( \det = 0 \Rightarrow \dim \ker = 1 \)

Observe: Can artificially make 2 new eqns
\[ a_0 x + a_1 x^2 + a_2 x^3 \]
\[ b_0 x + b_1 x^2 + b_2 x^3 \]

Now we have 4 (in eqns in the “variables"
\[ x, x^2, x^3 \]
Take a vector in null space of Sylv. with
First entry 1
\[ \begin{pmatrix} a_0 & a_1 & a_2 & 0 \\ 0 & a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 & 0 \\ 0 & b_0 & b_1 & b_2 \end{pmatrix} \]
\[ \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \end{pmatrix} \]
Note: \( (1, x, x^2) \) solves 1st eqn
\( (x, x^2, x^3) \) solves second.
Prop. \( \text{Res}(f, g) = 0 \iff f, g \text{ have common root.} \)

If. Say \( \alpha \) = root of \( f \) & \( g \).

\( \exists \) polys \( f_i, g_i \) of deg \( m - 1, n - 1 \) s.t.

\[
\begin{align*}
    f(x) &= (x - \alpha) f_i(x) \\
    g(x) &= (x - \alpha) g_i(x)
\end{align*}
\]

\( \implies f(x) g_i(x) - g(x) f_i(x) = 0. \)

Both terms have deg \( m + n - 1 \).

Equating coeffs to 0 gives \( m + n \) lin eqns in \( m + n \) vars.

(coefs of \( f_i, g_i \))

The matrix is the Sylv. matrix.

The existence of a soln shows \( \text{Res}(f, g) = 0 \).

Conversely, say \( \text{Res}(f, g) = 0 \). As above get soln \( f_i, g_i \) to

\[
    f(x) g_i(x) - g(x) f_i(x) = 0.
\]

A root \( \alpha \) of \( f \) must be a root of \( g \) or \( f_i \). If \( g \), done.

If \( f_i \), cancel \((x - \alpha)\) from \( f \) & \( f_i \) continue inductively.

[\( \square \)]
In proj space

\[ C = \mathbb{Z}(f) \]
\[ D = \mathbb{Z}(g) \]

Assume WLOG \([0:0:1]\)
on neither:

(\exists \text{ purely } \mathbb{Z} \text{ term})

\[ f(x,y,z) = z^m + a_{m-1} z^{m-1} + \ldots + a_0 \]
\[ g(x,y,z) = z^n + \ldots + b_0 \]
\[ a_i, b_i : \text{homog polys in } x,y \]
\[ \text{of deg } m-i, n-i \]

\[ \rightsquigarrow \text{ } R(x,y) \text{ resultant wrt } z \]
\[ \text{poly in } x,y. \]

Prop. \[ R(x,y) \] either \(= 0 \)
or \(\deg mn. \)

Example. \[ f(x,y,z) = x^2 + y^2 - z^2 \]
\[ g(x,y,z) = x^3 - x^2 z - x z^2 \]

\[ \rightsquigarrow \text{ } R(x,y) = -x^2 y^4 \rightarrow x = 0 \]
\[ 2 \text{ roots w/ mult} \]
\[ 4 \text{ roots w/ mult.} \]
\[ x = 0 : y^2 - z^2 = 0 \rightarrow [0:1:1],[0:1:-1] \]
both have mult 2 \(\rightarrow \)
\[ y = 0 : x^2 - z^2 = 0 \rightarrow [1:0:1],[1:0:-1] \]

or \( y = 0 \)
Resultants

\[ f(x) = a_0 + a_1 x + \cdots + a_m x^m \]
\[ g(x) = b_0 + \cdots + b_n x^n \]

\[ \to \text{ Sylvester matrix} \]

\[
\begin{pmatrix}
  a_0 & \cdots & a_m \\
  \vdots & \ddots & \vdots \\
  b_0 & \cdots & b_n
\end{pmatrix}
\]

n times

m times

det. is \( \text{Res}(f,g) \).

Prop. \( \text{Res}(f,g) = 0 \iff \text{common factor}. \)

Lemma. \( f, g \) have common factor \( \iff \)

\[ \exists s, t: \deg s < \deg g \]
\[ \deg t < \deg f \]

\[ fs + gt = 0. \]

Pf. \( \Rightarrow \) \( f, g \) have common factor

\[ \Rightarrow f = h f_1 \quad g = h g_1 \]
\[ \Rightarrow f g_1 - g f_1 = 0. \]

\( \Leftarrow \) \( fs + gt = 0. \) Assume no comm. fac.

\[ \Rightarrow \text{roots of } f \text{ are roots of } gt \]
\[ \Rightarrow \text{roots of } f \text{ are roots of } t \]
but \( \deg t < \deg f. \) \( \Box \)
Prop. \( \text{Res}(f, g) = 0 \iff \text{common factor} \)

Lemma. \( f, g \) have common factor \( \iff \)

\[ \exists \ s, t : \quad \deg s < \deg g \]
\[ \deg t < \deg f \]
\[ f s + g t = 0. \]

Pf of Prop. Want to know existence of \( s, t \) as in Lemma.

Let \( P(x) = f(x) s(x) + g(x) t(x) \)

\[ s = \sum_{i=0}^{\text{deg } f} a_i x^i \quad t = \sum_{i=0}^{\text{deg } g} b_i x^i \]

\[ P(x) = \left( a_m s_{n-1} + b_{m-1} t_{m-1} \right) x^{m+n-1} + \ldots \]

Solving for \( (s_0, \ldots, s_{n-1}, t_0, \ldots, t_{m-1}) \) or:

\[
\begin{pmatrix}
\vdots \\
\vdots \\
\end{pmatrix}
\begin{pmatrix}
a_0 & \cdots & a_m \\
\vdots & \ddots & \vdots \\
\vdots & \cdots & \vdots \\
\end{pmatrix}
\begin{pmatrix}
\vdots \\
\vdots \\
\end{pmatrix} = 0
\]

\[ b_0 \ldots b_n \]

\[ \square \]
Curves in $\mathbb{P}^2$ version

$C = Z(f)$, $D = Z(g)$

WLOG $[0:0:1] \notin C \cup D$,

$\Rightarrow f, g$ have $\mathbb{Z}$-only term,

which must be lead term.

$\Rightarrow f = a_0Z^m + a_1Z^{m-1} + \ldots$

$g = b_0Z^n + b_1Z^{n-1} + \ldots$

$a_i, b_i$ homog. deg $i$

in $x, y$.

$\Rightarrow R(x,y) = \text{Res}(f,g)$ poly in $x, y$.

Prop. $R(x,y)$ is either: $\equiv 0$ or homog. of deg $mn$.

If. To show $R(tx, ty) = t^{mn}R(x,y)$

$R(tx, ty) = \begin{pmatrix}
    a_0 & ta_1 & t^2 a_2 & \ldots \\
    ta_0 & t^2 a_1 & t^3 a_2 & \ldots \\
    b_0 & tb_1 & t^2 b_2 & \ldots
\end{pmatrix}$

Mult. rows by $t^0, t^1, \ldots, t^{n-1}, t^0, \ldots, t^{m-1}$

factor: $n(n-1)/2 + m(m-1)/2$

Divide cols by $t^0, \ldots, t^{m+n-1} \sim \text{Res}(f,g)$

factor: $(m+n)(m+n-1)/2$

Difference is $mn$. □
Bézout's Thm: \( C, D \subseteq \mathbb{P}^2 \) curves of deg \( m, n \) w/ no common irreducible components. Then they intersect \( mn \) times with multiplicity.

**Proof Setup**

1. Suffices to consider \( C, D \) irreducible.
2. \( \dim \text{CnD} = 0 \) \( \implies |\text{CnD}| < \infty \).
3. Without loss of generality (WLOG), change coordinates so \( x \neq 0 \) at all pts of \( \text{CnD} \).

4. Say \( C = \mathbb{Z}(f), \ D = \mathbb{Z}(g) \)

   \( \implies \mathbb{R}(x, y) \)

   **Step 1.** \( \mathbb{R}(x, y) \) homogeneous of deg \( mn \)

   If \( \mathbb{R}(x, y) = 0 \) then \( V[a:b] \in \mathbb{P}^1 \)

   \( f, g \) have common 0, violating \( 2\).

   **Apply Prop.**

   Write \( \mathbb{R}(x, y) = x^m \mathbb{R}_*(y/x) \) where

   \( \mathbb{R}_* \) is poly in \( t = y/x \) of deg \( \leq mn \).

   **Step 2.** deg \( \mathbb{R}_* = mn \).

   deg \( \mathbb{R}_* < mn \iff \) no \( y^{mn} \) term. \iff all terms of \( R \) have \( x \iff \mathbb{R}(0,1) = 0 \) violates \( 3\).
Step 3: Roots of \( R_x \)

\[ \iff \text{CND \ w/mult.} \]

\[ \text{If } \alpha \text{ is a root of } R_x \]

\[ \text{then } \alpha = a/b \text{ with } R(a,b) = 0. \]

\[ \Rightarrow f(a,b,z), g(a,b,z) \text{ have} \]

\[ \text{common root } \iff \text{pt of CND of form } [a:b:c]. \]

\[ \Rightarrow [a:b:c] \in \text{CND } a \neq 0 \]

\[ \Rightarrow \text{bla a root of } R_x \]

Define multiplicity now:

\[ \# \text{common roots } c \iff \text{mult.} \]

\[ \text{corresp. to given } \alpha. \]

\[ \text{or } \deg \text{ of } c \text{ as root of } \]

\[ \text{more prec. } f-g \text{ @ (a,b)}. \]

Claim: This equals \( \deg \text{ of } a \) as root of \( R_x \).

\[ \text{Mult. defined as mult of root of } R_x. \]

Need Setup 5: No two pts \( [a:b:c], [a:b:c'] \in \text{CND}. \)

\[ \iff \text{no pts of CND lie on line } || \text{ to } z\text{-axis}. \]
Let \( p \in \text{CND} \).

Assume \( p \) in std aff. chart \( z = 1 \).

Define

\[
\iota(\text{CND}, p) = \dim_k \left( \frac{\mathcal{O}_p}{(f,g)_p} \right)
\]

\( \mathcal{O}_p \): rat'l fns def. at \( p \).

\( (f,g)_p \): ideal gen by \( f,g \) in \( \mathcal{O}_p \).

“localization”: allow denominators that don’t vanish at \( p \).

i.e. denoms don’t lie max ideal at \( p \) which is \( (x_1-p_1), \ldots, (x_n-p_n) \).

**Example 1.** \( k = \mathbb{C} \)

\[
f(x,y) = y - x^3 \\
g(x,y) = y
\]

\[
\mathcal{O}_p = \mathbb{C}[x,y]_{(x,y)} \cong \left( \frac{\mathbb{C}[x,y]}{(y-x^3, y)} \right)_{(x,y)} \\
\cong \left( \mathbb{C}[x]/(x^3) \right)_{(x)} \cong \mathbb{C}^3 \text{ as } \mathbb{C} \text{- v.s.}
\]

**Example 2.** basis \( 1, x, x^2 \)

\[
x^3 = y \\
y = 0.
\]
Why are the multiplicities the same?

Write $I_p(C,D)$.

or $I_p(f,g)$

Fulton

Gims (?)

Axioms

1. $I_p(f,g) = I_p(g,f)$

2. $I_p(f,g) = \begin{cases} 0 & p \notin C \cap D \\ \infty & p \in \text{a common comp.} \end{cases}$

3. $C, D$ lines, $p \in C \cap D \Rightarrow I_p(f,g) = 1$

4. $I_p(f_1, f_2, g) = I_p(f_1, g) + I_p(f_2, g)$

5. $I_p(f,g) = I_p(f, g + fh)$ if $\deg h = \deg g - \deg f$.

Thm. An $I_p(f,g)$ satisfying the axioms exists and is unique.
Final HW

7 problems on web site or project on Teams 1-2 "pages"

Learn something & tell us about it.

Ideas: Robotics Splines Poncelet’s porism

Gröbner bases Image deblurring Hironaka’s thm Sheaves / Schemes

Divisors Riemann-Roch Thm or creative writing artwork.

Fano varieties
Computing multiplicities

\[ C = \mathbb{Z}(x^2+y^2-z^2) \]
\[ D = \mathbb{Z}(x^2+y^2-2z^2) \]
\[ C \cap D = [\pm i : 1 : 0] \]

Via axioms

\[ I_p(x^2+y^2-z^2, x^2+y^2-2z^2) \]
\[ = I_p(x^2+y^2-z^2, z^2) \quad \text{“row op”} \]
\[ = I_p(x^2+y^2, z^2) \]
\[ = 2I_p(x^2+y^2, z) \]
\[ = 2I_p(x+iy, z) + 2I_p(x-iy, z) \]
\[ = 2 + 0 \quad \text{or} \quad 0 + 2 \quad \text{“lines”} \]
\[ \text{dep. on } p \]

Via resultant

\[ R(x, y) = \det \begin{pmatrix} -1 & 0 & x^2+y^2 & 0 \\ 0 & -1 & 0 & x^2+y^2 \\ 0 & 0 & 0 & x^2+y^2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]
\[ = (x^2+y^2)^2 \]
\[ \sim R_*(t) = (1+t^2)^2 \]
\[ = (1+it)^2(1-it)^2 \]
via local rings

\[ f = \mathbb{Z}(y-x^2) \quad g = \mathbb{Z}(y) \]

\[ \mathbb{C}[x,y]/(f,g)(x,y) \]

\[ \cong (\mathbb{C}[x,y]/(f,g))(x,y) \]

\[ \{ \frac{ax+b}{cx+d} : d \neq 0 \} \]

WTS: \[ \dim = 2 \]

basis 1, x.

rationalize:

\[ \frac{ax+b}{cx+d} \quad \frac{-cx+d}{-cx+d} \]

\[ = -\frac{acx^2 + (ad-bc)x + bd}{d^2 - c^2 x^2} \]

\[ = \left( \frac{ad-bc}{d^2} \right)x + \frac{b}{d} \]

exercise. Do example on last slide this way.

easier. Fitchett example \[ x^3 = y, y = 0. \]
Easy conseq's of Bézout

1. If \(|C \cap D| = mn\), all intersections are transverse (mult 1)
2. If \(|C \cap D| > mn\), common irred. comp.
3. Any two proj. curves intersect.
4. \(|C \cap L| = m\) with mult. (L line).

More conseq's

5. \(C\) irred has at most \(\binom{d-1}{2}\) sing pts.
   Gathmann Prop 13.5
   Fulger Cor 8.14
   \(d = \deg C\)

6. Degree genus Formula.
   \(C\) smooth \(\Rightarrow g = \binom{d-1}{2}\)
   Kerr Sec 14.3.

\[\text{really}\]
7. Pascal's mystic hexagon.
   If a hexagon is inscribed in an irreducible conic then opposite sides meet in collinear points.

8. Cayley-Bacharach Thm
   Any cubic passing thru 8 pts, passes thru the 9th.

9. Space of smooth curves of degree $d$ in $\mathbb{P}^2$ is a complex projective space of dimension $d(d+3)/2$.

10. Harnack's thm
    A smooth real projective curve in $\mathbb{P}^2$ has at most $(d-1)+1$ loops.
11. A nonsing cubic has 9 pts of inflection

12. \( \text{Aut } \mathbb{P}^n \cong \text{GL}_{n+1} k \). Gathmann

13. Classific. of irreducible singular cubics:
   \[
   y^2 - x^3 - x^2 \\
   y^2 - x^3 \\
   y^2 - x^3
   \]

   and smooth cubics:
   \[
   y^2 = 4x^3 - ax - b
   \]

Hulek Prop 4.11

14. Smooth cubics are groups!

Dolgachev p.52
Mystic Hexagon

Prop. Say $C, D$ deg $n$

$|C \cap D| = n^2$

Assume $mn$ of the int pts lie on irred curve $E$ of deg $m$

Then the remaining $n(n-m)$ pts lie on a curve $F$

of $\leq n-m$.

PF of Mystic Hex

Say vertices are $p_0, \ldots, p_5$

Let $L_i = \text{line through } p_i p_{i+1} \pmod{5}$

Let $L = L_0 L_2 L_4$ $L' = L_1 L_3 L_5$ cubics.

$L$ & $L'$ have no common factor.

Bézout $\Rightarrow |L \cap L'| \leq 9$.

If $< 9$, nothing to do.

6 pts of $L \cap L'$ are $p_0, \ldots, p_5$

which lie on conic.

The other 3 lie on a line by the Prep.

Other pf Shafarevich
Prop. Say $C, D$ deg $n$
$|C \cap D| = n^2$
Assume $mn$ of the int pts
lie on irred curve $E$ of deg $m$
Then the remaining $n(n-m)$ pts
lie on a curve $F$
of deg $\leq n-m$.

Pf. Say $C, D, E$ given by $f, g, h$.
Let $[a:b:c] \in E \setminus (C \cap D)$
Let $F_0 = Z(p)$
$p = g(a,b,c)f - f(a,b,c)g$
deg $\leq n$.

Then $|E \cap F_0| \geq mn+1$ ble it
contains $mn$ pts of $E \cap (C \cap D)$
and $[a:b:c]$.
Bézout $\Rightarrow F_0, E$ have common comp.
$E$ irred, so it is a comp. of $F_0$
$\Rightarrow p = h q^E$ deg $q \leq n-m$
Let $F = Z(q)$
Each $[u:v:w]$ in $(C \cap D) \setminus E$ satisfies
$f=0, g=0$ thus satisfies $p=0$
Also $h(u,v,w) \neq 0 \Rightarrow q(u,v,w) = 0$,
i.e. $[u:v:w] \notin F$. □
Curves thru given pts

Existence

1. Thru 2 pts F line.
   \[ ax+by = c \]
   2 constraints (given pts)
   3 unknowns \( a, b, c \).

2. Thru 5 pts F quadric
   (same lin. alg)

   Bézout \( \Rightarrow \) if 3 are collinear
   then quadric is reducible
   (not a conic) \( \Rightarrow \) union of
   2 lines.

Uniqueness

Need some kind of general posn.

1. always unique (if 2 pts distinct)
2. if all 5 pts collinear, then can take that line & any other line.

But if no 3 collinear get uniqueness: can't be 2 distinct lines by hyp.

\( 3 \) Thru 9 pts F cubic (same lin alg).
If 8 of 9 pts lie on conic C then many CUL are cubics containing the 9 pts.

Even without this, uniqueness harder to come by.

Say $C_0 = \mathbb{Z}(f_0)$ cubics.

$C_{oo} = \mathbb{Z}(f_{oo})$

& $|C_0 \cap C_{oo}| = 9$.

Then $C_t = \mathbb{Z}(f_0 + tf_{oo})$ for $u \in \mathbb{K}$ contains all 9 pts.

But these are only ones going through the 9 pts, or even 8 of them.

**Cayley-Bacharach** thru $k$ alg closed.

If $D$ is a cubic curve passing thru $8$ pts of $C_0 \cap C_{oo}$ then $D = C_t$ for some $t$. In particular, $D$ passes thru the $9^{th}$ pt.
Claim 1. No 4 of the $a_i$ collinear.

Proof (Pf). Bézout $\Rightarrow$ $C_0, C_\infty$ would both contain this line.

$\Rightarrow |C_0 \cap C_\infty| = \infty > 9$.

Claim 2. No 7 of the $a_i$ lie on quadric.

Proof (Pf). Same.

Claim 3. Any 5 of the $a_i$ determine a unique quadric.

Proof (Pf). If 5 pts lie on two quadrics $E, F$.

Bézout $\Rightarrow E \cap F$ contains line $L$.

Claim 1 $\Rightarrow L$ contains at most 3 of the $a_i$.

The other $\geq 2$ pts must lie on other comp of $E$ (line)
& other comp of $F$ (line).

Both are lines. Must be same line. So $E = F$. 

**Claim:** Assume $D$ passing thru 8 pts $a_1, ..., a_8$ of $C_0 \cap C_\infty$.

Say $C_0 = Z(p_0), C_\infty = Z(p_\infty)$

$D = Z(p)$  Want: $D : C_t$

Assume $D \neq C_t$ any t.  

Claim 1. No 4 of the $a_i$ collinear.

Proof (Pf). Bézout $\Rightarrow C_0, C_\infty$ would both contain this line.

$\Rightarrow |C_0 \cap C_\infty| = \infty > 9$.

Claim 2. No 7 of the $a_i$ lie on quadric.

Proof (Pf). Same.
Claim 4. No 3 of the ai collin.

Proof. Say ai, a2, a3 \in L line.

Claim 1 \implies a_i \not\in L i > 3.

a2, ..., a8 lie on unique quadric E.

(Claim 3)

Let b be another pt on L

Let c be another pt not on E or L.

By lin alg E cubic

q = xp + yp + zp0

vanishing at b, c. (3 vars
2 eqns)

By \bigotimes q \neq 0.

Let F = \mathbb{Z}(q).

FnL contains a1, a2, a3, b

Bezout \implies F = L \cup quadric

The quadric contains

a4, ..., a8 (p, p0, p00 all

vanish at

a1, ..., a8)

By uniqueness of E:

F = L \cup E

but c not in E, L hence

not in F. contradiction

\square
Claim 5. No 6 of $a_1, \ldots, a_8$ lie on a quadric.

**Pf.** Say $a_1, \ldots, a_6$ lie on $Q = \text{quadric}$.

Claim 4 $\Rightarrow$ $Q \neq L_1 \cup L_2$

$\Rightarrow$ $Q$ conic.

Let $L$ = line thru $a_7, a_8$.

$b$ = another pt on $Q$

c = pt not on $L$ or $Q$

As before, have nonzero cubic $q = xp + yp_0 + zp_{00}$ vanishes at $b, c$. Also at $a_1, \ldots, a_8$

Let $F = \mathbb{Z}(q)$ Note $b, c \in F$. $F$ contains $a_1, \ldots, a_6, b \in Q$

$\Rightarrow F = Q \cup$ line

The line is $L$, but $L$ hence $F$ does not contain $c$.

Contrad.
Finishing...

Let \( L = \text{line thru } a_1, a_2 \)
\[ Q = \text{quadric thru } a_3, \ldots, a_7 \]

Claim 3 \( \Rightarrow \) \( Q \) unique

Claim 4 \( \Rightarrow \) \( Q \) cubic (can't be 2 lines)

Claim 4 \( \Rightarrow \) \( a_8 \notin L \)

Claim 5 \( \Rightarrow \) \( a_8 \notin Q \)

Let \( b, c \in L \setminus Q \)

Again, \( J \) non-0 cubic

\[ q = xp + yp_0 + zp_0 \]

Vanishing on \( b, c \Rightarrow F = Z(q) \)

\( F \cap L \) contains \( a_1, a_2, b, c \)

Bézout \( \Rightarrow F = L \cup Q \quad \text{quadric} \)

The quadric contains \( a_3, \ldots, a_7 \)

So it is \( Q \)

So \( F = L \cup Q \)

\( a_8 \) not in \( F \).

But \( F \) is a lin interp. of 3 cubics cont. \( a_1, \ldots, a_8 \).

\( \Box \)
Proof of Pappus

C, D are cubics given by triples of lines in hexagon.

E given by $L_1, L_2,$ line thru $c_1, c_2$

Cayley-Bacharach \( \Rightarrow \) $E$ contains $c_3$

(We assumed $c_1 \neq c_2$, o/w nothing to prove.)

Pascal’s Mystic Hexagon similar

(note: the $c_i$ can’t all lie on the conic by Bezout.)
Smooth cubics are groups

$C = \mathbb{Z}(y^2 - x^3 - ax - b) \subseteq \mathbb{P}^2$

smooth

$O = [0:1:0] \in C$

pt at $\infty$

For $c = [u:v:w]$

let $\overline{c} = [u:-v:w]$

refl. thru $x$-axis.
in $\mathbb{A}^2$ plane

so $\overline{O} = O$.

Define $a + b = \overline{c}$

\begin{align*}
\text{Thm. } C & \text{ is an abel. gp.} \\
\text{pf. } \text{identity: } & 0. \\
\text{inverse: } & c + \overline{c} = 0. \\
\text{abelian: } & \checkmark
\end{align*}
associativity, assume WLOG 
0, a, b, c, a+b, b+c, -(a+b), 
-(b+c) all distinct from 
each other and 
-(a+b)+c) & -(a+(b+c))

(uses smoothness)

Let \( D = \overline{ab}, \overline{c(a+b)}, \overline{o(b+c)} \)

\( E = \overline{o a+b} \overline{bc} = \overline{a(b+c)} \)

C & D cubics meeting at 9 pts;
no common comp.
E passes thru 8 hence 9th by Cay-Ba

The 9th pt is 
-(a+b)+c)
The line thru a, b+c meets C 
in \(-(a+(b+c)) \& -(a+b)+c) 
hence equal. \( \square \)
Tao says: Pascal is a degenerate case of the associativity law on cubic.

Pappus is a degenerate case of Pascal.

Mordell’s Thm: \( \mathbb{Q} \) pts on \( C \) form a fin. gen. abel. gp.
Goal: Classify cubic curves.

i.e. $C = \mathcal{Z}(f) \subseteq \mathbb{P}^2$

$\deg 3 \quad (\text{char } k = 0)$

proj. equiv: $\text{GL}_3 k$

4 cases: 1. 3 lines
2. conic + line
3. sing irred
4. smooth irred cubic

Case 1 3 lines.

lines in $\mathbb{P}^2 \leftrightarrow$ pts in $\mathbb{P}^2$

via orthog. compl. in $k^3$

Prop. $C =$ union of 3 lines

Then $C$ is proj eq to exactly one of

1. $\mathcal{Z}(xyz)$
2. $\mathcal{Z}(x^4(x+y))$
3. $\mathcal{Z}(x^2y + y^2z + z^2x)$
4. $\mathcal{Z}(x^3 + y^3 + z^3)$

$\text{PF. Translate to problem about pts in } \mathbb{P}^2$

$1 \iff$ collinear, distinct...
Case 2: Conic + line

Prop. $C = \text{conic} + \text{line} = Q \cup L$

The $C$ is proj eq to exactly one of

1. $\mathbb{P}^1(\mathbb{C})$ 
2. $\mathbb{P}^2(\mathbb{C})$

Proof. We already showed (using quad. forms) $Q$ is proj equiv to $\mathbb{P}^1(\mathbb{C})$ and $L$ is hence determined.

Bézout $\rightarrow$ 2 cases

1. $\lvert Q \cap L \rvert = 2$
2. $\lvert Q \cap L \rvert = 1$

$Q$ is image of $\mathbb{P}' \rightarrow \mathbb{P}^2$

Up to change of coords in $\mathbb{P}^1$ can assume int. pts are

1. $[1:0:0]$ & $[0:0:1]$
2. $[0:0:1]$

Show $\exists$ linear change of coords on $\mathbb{P}^2$ realizing this reparameterization.
If the param of $Q$ is $[t : u] \mapsto [t^2 : tu : u^2]$.

If the parametrization in $\mathbb{P}^1$ is $(a \ b)$, then in $\mathbb{P}^2$ the reparam. is
\[
\begin{pmatrix}
    a^2 & 2ab & b^2 \\
    ac & ad+bc & bd \\
    c^2 & 2cd & d^2
\end{pmatrix}
\]

Case 3: Sing. irred. cubics.

**Prop.** $C = \text{sing. irred. cubic}$.

Then $C$ is proj equiv to exactly one of

1. $\mathbb{Z}(y^2z-x^3-x^2z)$
2. $\mathbb{Z}(y^2z-x^3)$
Fact. \( X = \mathbb{Z}(f) \subseteq \mathbb{P}^2 \)
\[ p \in X \quad L = \text{line} \]
Then \( I_p(X,L) = \text{mult}_p(f|L) \)

Proof. Change coords so \( p = [0:0:1] \)
& \( L = \mathbb{Z}(y) \)
Let \( \overline{f}(x) = f(x,0,1) \)
\[ I_p(X,L) = \dim \mathcal{O}_{\mathbb{P}^2, [0:0:1]} / (f,y)_{[0:0:1]} \]
\[ = \dim \mathcal{O}_L, (0,0) / (f,y)_{(0,0)} \]
\[ = \dim (k[x,y] / (f,y))_{(0,0)} \]
\[ = \dim (k[x,y] / (f))_1 \] is smallest degree of a term of \( \overline{f} \)

Example.
\[ f(x,y,1) = x^3y + x^2y^2 + x^2 + x^3 \]
\[ \overline{f}(x) = x^2 \]

Cor 1. \( X = \mathbb{Z}(f) \subseteq \mathbb{P}^2 \), \( p \in X \)
TFAE
1. \( \text{mult}_p(f|L) > 1 \)
2. \( L \subset T_p X \)
3. \( I_p(X,L) > 1 \)

Proof. We already \( 1 \Leftrightarrow 2 \)
Fact gives \( 1 \Leftrightarrow 3 \)
Cor 2. $C \subseteq \mathbb{P}^2$ cubic curve. Then $C$ has at most 1 sing pt.

PF. Suppose $p, q$ singular, $p \neq q$.

Let $L = \overline{pq}$ (line)

$T_p C \cong T_q C \cong \mathbb{A}^2$

Cor 1 $\Rightarrow I_p (C, L) \geq 2$

$I_q (C, L) \geq 2$

Contradicts Bezout.
Proof of Case 3 Prop

Assume the sing. is at \([0:0:1]\)

\[ f = bx^3 + cx^2y + dx^2y + ey^3 + q(x,y) \]

\[ q(x,y) = \text{quad form in } x, y. \]

(Since \((0,0) \in C\), no const. term.
Since \((0,0)\) sing., no linear terms.)

Have \(q(x,y) \neq 0\) because then

\( f \) factors into product of 3 linear.
(divide by \(y^3 \rightarrow \)

\( \text{poly of deg 3 in } x/y \).)

Can factor \( q(x,y) = l_0(x,y)L_1(x,y) \)

Case 1. \( l_0, l_1 \) not multiples

Case 2. \( l_0 = cL_1 \) (multiples).

Clever change of vars.

e.g. in Case 2, wlog \( l_0 = l_1 = y \)

\[ f = bx^3 + cx^2y + dx^2y + ey^3 + y^2 \]

(linear)
Change of vars: \( x = x' - \frac{c}{3b} y \)

gets rid of \( x^2y \) term

etc... \( \square \)
Case 4 Smooth irreducible cubics.

Prop. \( C \) smooth irr cubic

Then \( C \) is equiv to some \( C_{b,c} = \mathbb{Z}(f_{b,c}) \) Weierstrass curves.

\[
f_{b,c} = y^2 - 4x^3 + bx + c
\]

\[ H \] flex pts & Hessians

\( p \in C \) is a flex pt (or inflection pt)

if \( I_p(C, T_p C) \geq 3 \)

If \( C = \mathbb{Z}(f) \subseteq \mathbb{P}^2 \)

\[
H_f = \det \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{0 \leq i,j \leq 2}
\]

Have: \( H_f \) is \( \equiv 0 \) or homog of deg \( 3(d-2) \)

\[ H \] Hessian curve \( H \subseteq \mathbb{P}^2 \)

Prop. \( H \cap C = \{ \text{flex pts of } C \} \)

Cor. \( C \) has a flex pt.
Discriminants

Define \( \text{Disc}(f_b,c) \) to be \( b^3 - 27c^2 \)

Fact. If \( \alpha_i \) are roots of \( f \),
\[ \text{Disc}(f_b,c) = \alpha_n^{2n-2} \prod_{i \neq j} (\alpha_i - \alpha_j) \]

Define \( \text{Disc}(C_b,c) = \text{Disc}(f_b,c)/16 \)

Prop. \( C_b,c \) smooth \( \iff \) \( \text{Disc}(C_b,c) \neq 0 \).

Proof of Prop. Let \( p = \text{flex pt of } C \)

WLOG \( p = [0:0:1] \)

\& \( T_pC = \mathbb{Z}(x) = L \)

\[ \implies f|_L \text{ has } 0 \text{ of order } 3 \text{ at } 0. \]

\[ \implies f = -y^3 + x(ax^2 + by^2 + cz^2 + dxz + eyz + gyz) \]

No quadratic terms (flex pt)

Plugging in \( x = 0 \) needs to give deg 3 in \( y \).

Clever change \( p \) smooth \( \implies c \neq 0 \) of coords \( \Box \)
Smooth cubic curves

Last time: every smooth irreducible cubic in \( \mathbb{P}^2 \) is proj. equiv to \( C_{b,c} = Z(f_{b,c}) \)

\[
f_{b,c} = y^2 - 4x^3 + bx + c.
\]

Also: \( C_{b,c} \) smooth \( \iff \) \( \text{Disc}(f_{a,b}) \neq 0 \)

\[
b^3 - 27c^2
\]

Conseq. \( \{ \text{Smooth } C_{b,c} \} \) is \( \cong \) a.a.v.

---

J - invt

\[
J: \{ C_{b,c} \} \rightarrow C
\]

\[
C_{b,c} \mapsto \frac{b^3}{b^3 - 27c^2}
\]

Equiv. reln on \( \{ C_{b,c} \} \): differ by proj aut fixing \([0:0:1] \).

Prop. \( C_{b,c} \sim C_{b',c'} \iff \) same \( \chi \) (smooth)

Lemma. Any proj aut. fixing \([0:0:1] \) is of form

\[
\begin{align*}
\chi & \mapsto u^2 x \\
y & \mapsto u^3 y
\end{align*}
\]

Pf. (in alg...
Prop. \( C_{b,c} \sim C_{b',c'} \iff \text{same} \)
(smooth)

Pf. Special case \( J = 0 \)
\[ \Rightarrow \text{easy using lemma.} \]
\[ \Leftarrow J = 0 \Rightarrow b = b' = 0, c \neq 0. \]
Choose \( u \) s.t. \( c' = c/u \).

By Prop:
\[ J: \{ \text{smooth } C_{b,c} \} \sim_C \]

Another point of view \( k = \mathbb{C} \).
Every \( C_{b,c} \) is homeo to \( \mathbb{C}^2 = T^2 \)
\[ y^2 = 4x^3 - bx - c = 4(x - \lambda_1)(x - \lambda_2)(x - \lambda_3) \]
\( C_{b,c} \) has an involution \( (y, x) \mapsto (-y, x) \)

\[ \mathbb{P}^2 \]
\[ \text{Cb,c} \]

Project 

Iviste 

\[ \text{quotient by } L \]

Upshot: \( C_{b,c} \cong T^2 \) as Riemann surf. (complex manifold)
Another way to make a torus

\( \omega_1, \omega_2 \in \mathbb{C} \sim \)

\[ \Lambda = \{ z \omega_1 + z \omega_2 \} \]

\[ E_\Lambda = \mathbb{C}/\Lambda \cong \mathbb{T}^2 \] "elliptic curve"

 Equivalence on \( \{ E_\Lambda \} \).

Given \( \Lambda \), can rotate, flip, scale so

\[ \omega_1 = 1 \]

\[ \text{Im} \omega_2 > 0 \quad (\omega_1 = 1, \omega_2 = \tau) \]

\[ E_\Lambda \cong E_\tau \quad \tau \in \text{upper half plane} \]

Moreover: \( \text{SL}_2 \mathbb{Z} \triangleleft \text{upper half-plane} \) by Möbius transform.

Fact: \( E_\tau \sim E_{\tau'} \iff \tau \sim \tau' \mod \text{SL}_2 \mathbb{Z} \).

equiv: biholomorphism.

Will show: \( \{ E_\Lambda \}_\sim \leftrightarrow \{ \text{smooth} \} / \sim \mathbb{C} \).
Fact: \( E_\tau \sim E_{\tau'} \iff \tau \sim \tau' \mod \text{SL}_2 \mathbb{Z} \).

Example: \( \tau = i \).

\[
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot i = \frac{1 \cdot i + 1}{0 \cdot i + 1} = 1 + 1 = \tau'
\]

Note: Mod. surf homeo to \( \mathbb{C} \)
Hexagonal torus

$e^{i\pi/3}$

Cut & paste into a reg. hexagon
Now have: \( \{ \text{smooth} \} \) \& \( \{ E^A \}/\sim \)
both homeo to \( C. \)

Want: Map between them.

\textbf{Weierstrass} \( \wp \) \textbf{function}

Assume \( \Lambda = \mathbb{Z} + \mathbb{Z} \tau \)

\( \wp(z) = \wp_\Lambda(z) = \)

\[ \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus 0} \left( \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right) \]

Invariant under \( \Lambda \), i.e.

it is a fn on \( E_\Lambda \)

Get a map:

\( \wp : E_\Lambda \to C_{b,c} \)

\[ z \mapsto \left[ 1 : \wp(z) : \wp'(z) \right] \]

where

\[ b = 60 \sum_{\omega \in \Lambda \setminus 0} \frac{1}{\omega^4} \]

\[ c = 140 \sum_{\omega \in \Lambda \setminus 0} \frac{1}{\omega^6} \]

Works because

\[ (\wp')^2 = 4 \wp^3 - b \wp - c. \]

This the desired map \( \{ E^A \}/\sim \to \{ \text{smooth} \} \)

\( \text{mod. surf.} \)

\( \text{Injectivity: J-inv.} \)

\( \text{Surj.: J is holom. nonconst. map} \)
Cayley-Salomon Thm: Every smooth cubic surface in $\mathbb{P}^3$ contains exactly 27 lines.

and the (non)-intersection pattern given by

**Clebsch**

\[ \text{(diagram showing lines and points)} \]
Basic strategy

1. Show that $Z(x^3 + y^3 + z^3 + w^3)$, "Fermat curve", has exactly 27 lines.

2. The number of lines is locally constant in moduli space of smooth cubic surfaces (which is connected).

In alg. top. language.

- Space of pairs $(x, L) \in X$
- Space of smooth cubics
- Deg 27 cov. space.
27 LINES

A cubic surf. is

$S = Z(f) \subseteq \mathbb{P}^3$

where $\deg f = 3$.

Cayley-Salmon Thm

$S$ smooth $\Rightarrow$

$S$ contains exactly

27 lines

Strategy. Show that some $S$ has 27 lines and

1) # lines is locally const.
   in space of smooth cubic surfaces.

2) The same $S$ is Fermat cubic:

$Z(x_0^3 + x_1^3 + x_2^3 + x_3^3)$
Lemma. The Fermat cubic $X$ has 27 lines (exactly)

$$X = \mathbb{Z} (x_0^3 + x_1^3 + x_2^3 + x_3^3)$$

If $X$ invt under permutation of coords.

Up to such permvt, any line is

$$x_0 = a_2 x_2 + a_3 x_3$$
$$x_1 = b_2 x_2 + b_3 x_3$$

(move the 2 pivots to left)

Such a line lies in $X$ $\iff$

$$0 = (a_2 x_2 + a_3 x_3)^3 + (b_2 x_2 + b_3 x_3)^3 + x_2^3 + x_3^3$$

Compare coeffs of $\text{LHS}=0$ & $\text{RHS}$

$$\Rightarrow a_2^3 + b_2^3 = -1 \quad (1)$$
$$\Rightarrow a_3^3 + b_3^3 = -1 \quad (2)$$
$$\Rightarrow a_2 a_3 = - b_2 b_3 \quad (3)$$
$$\Rightarrow a_2 a_3^2 = - b_2 b_3^2 \quad (4)$$

If $a_2, b_2, a_3, b_3, a_2, b_2, a_3, b_3$ all $\neq 0$ then $(3)^3/(4)$

$$\Rightarrow a_2^3 = - b_2^3 \quad \text{contradicting (1)}.$$

So WLOG $a_2 = 0$

$$\Rightarrow b_2^3 = -1 \quad (1)$$
$$\Rightarrow b_3^3 = 0 \quad (3)$$
$$\Rightarrow a_3^3 = -1 \quad (2)$$

$\Rightarrow 9$ lines ($3$ choices for each $3\sqrt{-1}$) $\blacksquare$
How are the lines related?

Intersection pattern: (complement of) Schlafli graph

Claim: Each of the 27 lines in a cubic surface intersects 10 of the others.

Idea: Given one line \( L \), consider the family of planes \( \{P_x\} \subseteq \mathbb{P}^3 \) containing \( L \).

\( P_x \cap S = \text{cubic curve } X_x \)

By our classification:

\( X_x = 3 \text{ lines } L, L', L'' \)

or \( L \cup \text{C}_x \text{ conic} \)

(need to rule out double lines).

\( \text{C}_x \text{ being } 2 \text{ lines or conic is a smoothness/Jacobian condition} \)

\( \rightarrow \deg 5 \text{ poly in } \lambda. \)

For each of the 5 roots, get 2 lines intersecting \( L \).
Moduli space of smooth cubic surfaces

All cubic surfaces:
\[ \mathbb{P}^1 = \mathbb{P}^{(3+3)} - 1 \]

3 balls in 4 boxes

Claim:
Smoothness for \( \mathbb{Z}(f) \)

\[ \iff \text{rk} \left( \frac{df}{dx_i} \right) \neq 0. \]

\[ \iff \text{rk} \geq 1 \]

\[ \iff \text{tangent space} \leq 2. \]

Lucky accident: The zeros of \( \frac{df}{dx_i} \) are all on \( \mathbb{Z}(f) \)

Why? Euler's identity

\[ 3f = \sum x_i \frac{df}{dx_i} \]

By claim: Moduli space of smooth cubic surfaces is complement of intersection of 4 hypersurfs in \( \mathbb{P}^{19} \).

\[ \rightarrow \text{dense} \& \text{open in } \mathbb{P}^{19} \]

\[ \Rightarrow \text{moduli sp. is connected (codim reasons)} \]
The Incidence Correspondence

Main idea: $M$ is locally const.

Two ways of rephrasing this:
1. $M \xrightarrow{p} U$ is a cov. sp.
   $(p^{-1}(x))$ is locally const.


WTS this # is indesp. of $X$.

Have: $\# \text{lines in } X = |\mathfrak{m}^{-1}(x)|$
\[ x \in X \implies \mathfrak{m}^{-1}(x) \neq \emptyset \]
\[
\begin{align*}
\Rightarrow & \quad \text{proj.}\ M \quad \xrightarrow{p} U \\
\phantom{\Rightarrow} & \quad (x, L) \quad \mapsto \quad x \\
\end{align*}
\]

There is projection $\pi : M \rightarrow U$.

$M = \mathfrak{m}(x, L) \subseteq U \times G(2, 3)$
\[ n \times U = M \subseteq G(2, 3) \]

Books like a graph.

$M$ looks like a graph.

$M$ is the graph of a continuously diff.

$U = \text{mod. sp. of sm. cub. surf}$

$G(2, 3)$
Blowing up $\mathbb{P}^2$

**Thm.** Every smooth cubic $S$ surface is the blowup of $\mathbb{P}^2$ at $6$ pts.

**Cor.** $S \cong \mathbb{CP}^2 \#_6 \mathbb{CP}^2$

$\implies \pi_1(S) = 1$

$\implies H_2(S) \cong \mathbb{Z}^7$

(intersection form type $(1,-6)$ etc.)

Idea of Thm

Further analysis of above work $L_1L_2$\implies $S$ has 2 disjoint lines.

(we found the ones that intersect)

Define map $\phi: L_1 \times L_2 \rightarrow S$

Works except when $xy$ is one of the 27 lines.

\[ \phi(x,y) \]

the 3rd pt on $\overline{xy}$ in $S$. 

\[ L_1 \]

\[ L_2 \]
Need to blow up \( L_1 \times L_2 \) in 5 pts to get well def map.

And: \( L_1 \times L_2 \cong \mathbb{P}' \times \mathbb{P}' \rightarrow \mathbb{P}^2 \)

\( \mathbb{P}' \times \mathbb{P}' \)

\( \cong \mathbb{P}^2 \) blown up at 1 pt.

(stereographic proj.)